These notes correspond to Section 2.8 in the text.

**The Existence-Uniqueness Theorem**

So far, we have learned how to solve certain specific types of first-order ODE. We now prove that under certain assumptions, a general first-order ODE has a unique solution.

**Theorem (Existence-Uniqueness)** Let the function \( f \) and \( \frac{\partial f}{\partial y} \) be continuous in some rectangle \( \alpha \leq t \leq \beta, \gamma \leq y \leq \delta \) containing the point \((t_0, y_0)\). Then, in some interval \( t_0 - h \leq t \leq t_0 + h \) contained in \((\alpha, \beta)\), the initial value problem

\[
y' = f(t, y), \quad y(t_0) = y_0
\]

has a unique solution \( y(t) \).

**Proof** We first note that if we integrate both sides of the equation \( y'(s) = f(s, y(s)) \) with respect to \( s \), from \( t_0 \) to \( t \), then we obtain

\[
y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds.
\]

Then, we define a sequence of functions, known as the *Picard iterates*, as follows:

\[
y_0(t) = y_0, \quad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s)) \, ds, \quad n = 0, 1, 2, \ldots
\]

Our goal is to prove that for some interval \( |t - t_0| \leq h \) contained within \([\alpha, \beta]\), the Picard iterates converge to a function \( y(t) \) that is a solution of the initial value problem. To that end, we first note that from the continuity of \( f \) on \( R \), we have

\[
|y_1(t) - y_0| = \left| \int_{t_0}^{t} f(s, y_0) \, ds \right| \leq M |t - t_0|,
\]

where \( |f(t, y_0)| \leq M \) on \([t_0 - h, t_0 + h]\).

Then, we let \( n \geq 1 \) and examine the difference

\[
|y_{n+1}(t) - y_n(t)| = \left| \int_{t_0}^{t} f(s, y_n(s)) - f(s, y_{n-1}(s)) \, ds \right| \leq \int_{t_0}^{t} |f(s, y_n(s)) - f(s, y_{n-1}(s))| \, ds.
\]

Because of the continuity of \( \frac{\partial f}{\partial y} \) in the rectangle \( R = [\alpha, \beta] \times [\gamma, \delta] \), we have, for \( a, b \in [\gamma, \delta] \) and \( |s - t_0| \leq h \), by the Mean Value Theorem,

\[
|f(s, b) - f(s, a)| = \left| \frac{\partial f}{\partial y}(s, \psi)(b - a) \right| \leq K |b - a|,
\]

where \( \psi \) lies between \( a \) and \( b \) and \( |\frac{\partial f}{\partial y}| \leq K \) on \( R \). It follows that

\[
|y_{n+1}(t) - y_n(t)| \leq K \left| \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)| \, ds \right|.
\]
Setting \( n = 1 \) yields

\[
|y_2(t) - y_1(t)| \leq K \left| \int_{t_0}^{t} |y_1(s) - y_0(s)| \, ds \right| \leq K M \left| \int_{t_0}^{t} |s - t_0| \, ds \right| \leq \frac{K M |t - t_0|^2}{2}.
\]

Continuing by induction, we obtain

\[
|y_{n+1}(t) - y_n(t)| \leq \frac{M K^n |t - t_0|^{n+1}}{(n + 1)!} \leq \frac{M K^n h^{n+1}}{(n + 1)!}.
\]

It follows that as \( n \to \infty \), \( |y_{n+1}(t) - y_n(t)| \to 0 \), thus establishing convergence of the Picard iterates to a solution

\[
y(t) = \lim_{n \to \infty} y_n(t).
\]

Now that we have proved that a solution exists, we show that it is also unique. We assume that there are two distinct solutions, \( y(t) \) and \( \tilde{y}(t) \). Then we have

\[
|y(t) - \tilde{y}(t)| = \left| \int_{t_0}^{t} f(s, y(s)) - f(s, \tilde{y}(s)) \, ds \right| \leq K \left| \int_{t_0}^{t} |y(s) - \tilde{y}(s)| \, ds \right|.
\]

Let \( z(t) = |y(t) - \tilde{y}(t)| \). Then, \( z(t) \geq 0 \), \( z(t_0) = 0 \), and for \( t \geq t_0 \), we have, by the Fundamental Theorem of Calculus,

\[
z'(t) \leq K |y(t) - \tilde{y}(t)| = K z(t).
\]

Rearranging and using the integrating factor \( \mu(t) = e^{-Kt} \), we obtain

\[
e^{-Kt} z'(t) - K e^{-Kt} z(t) = [e^{-Kt} z(t)]' \leq 0.
\]

Integration from \( t_0 \) to \( t \) yields \( e^{-Kt} z(t) \leq 0 \), and therefore \( z(t) \leq 0 \). However, we also have \( z(t) \geq 0 \). We conclude that \( z(t) = 0 \), which implies that \( y(t) = \tilde{y}(t) \), contradicting our assumption that these solutions are distinct. We conclude that the solution is unique. \( \Box \)

We now make two noteworthy observations about the preceding proof.

- The proof relies on the fundamental result that a continuous function on a compact set, such as a closed interval \([a, b]\) or a closed rectangle \([a, b] \times [c, d]\), has a maximum and minimum.

- The use of Picard iterates is an example of fixed-point iteration, in which an equation of the form \( g(x) = x \) is solved by choosing an initial guess \( x_0 \) and then computing \( x_{n+1} = g(x_n) \) for \( n = 0, 1, 2, \ldots \). This iteration provides both a practical and theoretical foundation for various methods of solving nonlinear equations or systems of equations.