These notes correspond to Section 3.2 in the text.

The Wronskian

Now that we know how to solve a linear second-order homogeneous ODE

\[ y'' + p(t)y' + q(t)y = 0 \]

in certain cases, we establish some theory about general equations of this form. First, we introduce some notation. We define a second-order linear differential operator \( L \) by

\[ L[y] = y'' + p(t)y' + q(t)y. \]

Then, a initial value problem with a second-order homogeneous linear ODE can be stated as

\[ L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = z_0. \]

We state a result concerning existence and uniqueness of solutions to such ODE, analogous to the Existence-Uniqueness Theorem for first-order ODE.

**Theorem (Existence-Uniqueness)** The initial value problem

\[ L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = z_0 \]

has a unique solution on an open interval \( I \) containing the point \( t_0 \) if \( p, q \) and \( g \) are continuous on \( I \). The solution is twice differentiable on \( I \).

Now, suppose that we have obtained two solutions \( y_1 \) and \( y_2 \) of the equation \( L[y] = 0 \). Then

\[ y_1'' + p(t)y_1' + q(t)y_1 = 0, \quad y_2'' + p(t)y_2' + q(t)y_2 = 0. \]

Let \( y = c_1y_1 + c_2y_2 \), where \( c_1 \) and \( c_2 \) are constants. Then

\[
L[y] = L[c_1y_1 + c_2y_2] = (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\
= c_1y_1'' + c_2y_2'' + c_1p(t)y_1' + c_2p(t)y_2' + c_1q(t)y_1 + c_2q(t)y_2 \\
= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\
= 0.
\]

We have just established the following theorem.

**Theorem (Superposition)** Let \( y_1 \) and \( y_2 \) be solutions of the equation \( L[y] = 0 \). Then, for any constants \( c_1 \) and \( c_2 \), the linear combination

\[ c_1y_1 + c_2y_2 \]

is also a solution.
Now that we can obtain infinitely many solutions of \( L[y] = 0 \) from two solutions \( y_1 \) and \( y_2 \), it is natural to ask whether all solutions of \( L[y] = 0 \) are of the form \( c_1 y_1 + c_2 y_2 \), for constants \( c_1 \) and \( c_2 \). For this to be the case, it is necessary to be able to satisfy any given initial conditions.

Let \( y(t) = c_1 y_1(t) + c_2 y_2(t) \). Substituting this solution into the initial conditions \( y(t_0) = y_0 \) and \( y'(t_0) = z_0 \), we obtain the system of equations

\[
\begin{align*}
    c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0, \\
    c_2 y_1'(t_0) + c_2 y_2'(t_0) &= z_0,
\end{align*}
\]

or, in matrix vector form,

\[
Y(y_1, y_2)(t_0) \mathbf{c} = \mathbf{u}_0,
\]

where

\[
Y(y_1, y_2)(t_0) = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \mathbf{u}_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.
\]

From linear algebra, this system has a unique solution for any right-hand side \( \mathbf{u}_0 \) if and only if the coefficient matrix \( Y(y_1, y_2)(t_0) \) has a nonzero determinant. That is, we must have

\[
W(y_1, y_2)(t_0) = \det Y(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0.
\]

The function \( W(y_1, y_2)(t) \), which is a function of \( t \) but depends on the solutions \( y_1(t) \) and \( y_2(t) \), is called the Wronskian of \( y_1 \) and \( y_2 \). If the Wronskian is nonzero, then we can satisfy any initial conditions. We have just established the following theorem.

**Theorem** Let \( y_1 \) and \( y_2 \) be two solutions of \( L[y] = 0 \). Then there exist constants \( c_1 \) and \( c_2 \) so that

\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

satisfies \( L[y] = 0 \) and the initial conditions

\[
y(t_0) = y_0, \quad y'(t_0) = z_0
\]

if and only if the Wronskian

\[
W = y_1 y_2' - y_2 y_1'
\]

is nonzero at \( t_0 \).

We can actually make a stronger statement: if the Wronskian is nonzero, then not only can we obtain a solution for any initial conditions, but we can actually describe all solutions of the initial value problem. That is, there are no other solutions that are not a linear combination of \( y_1 \) and \( y_2 \). This is formally stated in the following theorem.

**Theorem** Let \( y_1 \) and \( y_2 \) be solutions of \( L[y] = 0 \). Then every solution of \( L[y] = 0 \) is of the form

\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

if and only if the Wronskian of \( y_1 \) and \( y_2 \) is nonzero at a point \( t_0 \).

Because the linear combination

\[
y(t) = c_1 y_1(t) + c_2 y_2(t)
\]

describes all solutions of the equation \( L[y] = 0 \), it is called the general solution of this equation. We also say that the solutions \( y_1 \) and \( y_2 \) form a fundamental set of solutions of the equation.
Example Consider the ODE

\[ y'' + 4y' + 4y = 0. \]

Two solutions of this ODE are \( y_1(t) = e^{-2t} \) and \( y_2(t) = te^{-2t} \). Their Wronskian is

\[
W(y_1, y_2)(t) = e^{-2t}(te^{-2t})' - te^{-2t}(e^{-2t})' \\
= e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t}) \\
= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\
= e^{-4t},
\]

which is nonzero. Therefore, \( y_1 \) and \( y_2 \) form a fundamental set of solutions, and all solutions of the equation are of the form \( c_1y_1 + c_2y_2 \). □

Previously, when solving the constant-coefficient equation

\[ y'' + py' + qy = 0, \]

where the roots \( \lambda_1 \) and \( \lambda_2 \) of the characteristic equation \( \lambda^2 + p\lambda + q = 0 \) are real and distinct, we called the solution

\[ y(t) = c_1e^{\lambda_1t} + c_2e^{\lambda_2t} \]

the general solution. This is justified because at any time \( t \),

\[
W(e^{\lambda_1t}, e^{\lambda_2t})(t) = e^{\lambda_1t}(e^{\lambda_2t})' - e^{\lambda_2t}(e^{\lambda_1t})' \\
= \lambda_2e^{\lambda_1t}e^{\lambda_2t} - \lambda_1e^{\lambda_1t}e^{\lambda_2t} \\
= e^{\lambda_1t}e^{\lambda_2t}(\lambda_2 - \lambda_1) \\

\ne 0,
\]

so by the preceding theorems, \( y(t) \) actually is the general solution in the sense in which we have just defined it, and \( \{e^{\lambda_1t}, e^{\lambda_2t}\} \) is a fundamental set of solutions. If \( \lambda_1 = \lambda_2 \), however, we do not have a fundamental set of solutions, as the Wronskian would be zero. Later, we will learn how to obtain a second solution which, paired with \( e^{\lambda_1t} \), will form a fundamental set of solutions.

For the more general linear homogeneous second-order ODE, we can obtain a fundamental set of solutions by solving two specific initial value problems.

**Theorem** Let \( p(t) \) and \( q(t) \) be continuous on an open interval \( I \) containing a point \( t_0 \). Let \( y_1 \) be the unique solution of the ODE

\[ L[y] = y'' + p(t)y' + q(t)y = 0 \]

with initial conditions

\[ y(t_0) = 1, \quad y'(t_0) = 0; \]

and let \( y_2 \) be the unique solution of \( L[y] = 0 \) with initial conditions

\[ y(t_0) = 0, \quad y'(t_0) = 1. \]

Then \( y_1 \) and \( y_2 \) form a fundamental set of solutions of \( L[y] = 0 \).

To prove this theorem, we simply note that

\[
W(y_1, y_2)(t_0) = \det \left( \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \right) = \det \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 1(1) - 0(0) = 1 \neq 0,
\]
which proves that \( \{y_1(t), y_2(t)\} \) is a fundamental set of solutions.

**Example** Consider the ODE

\[ y'' - 3y' + 2y = 0. \]

Its characteristic equation is \( \lambda^2 - 3\lambda + 2 = 0 \), which has roots \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \). Therefore, the general solution is \( y(t) = c_1 e^t + c_2 e^{2t} \). From

\[
y(0) = c_1 + c_2, \quad y'(0) = c_1 + 2c_2,
\]

we find that the solution \( w_1(t) = 2e^t - e^{2t} \) satisfies the initial conditions

\[
w_1(0) = 1, \quad w_1'(0) = 0.
\]

Similarly, the solution \( w_2(t) = -e^t + e^{2t} \) satisfies the initial conditions

\[
w_2(0) = 0, \quad w_2'(0) = 1.
\]

Then \( \{w_1, w_2\} \) is a fundamental set of solutions of the ODE. \( \square \)

We conclude by deriving a simple formula for the Wronskian of any fundamental set of solutions \( \{y_1, y_2\} \) of \( L[y] = 0 \). Because they are solutions, we have

\[
y''_1 + p(t)y'_1 + q(t)y_1 = 0, \quad y''_2 + p(t)y'_2 + q(t)y_2 = 0.
\]

Multiplying the first equation by \( y_2 \) and the second equation by \( y_1 \), and then subtracting the first equation from the second, we obtain

\[
y''_2y_1 - y'_1y_2 + p(t)(y_1y'_2 - y_2y'_1) = 0.
\]

By noting that

\[
\frac{d}{dt}[W(y_1, y_2)(t)] = \frac{d}{dt}[y_1y'_2 - y_2y'_1] = y_1y''_2 + y'_1y'_2 - y'_2y'_1 - y_2y''_1 = y''_2y_1 - y''_1y_2,
\]

we obtain

\[
\frac{d}{dt}[W(y_1, y_2)(t)] + p(t)W(y_1, y_2)(t) = 0.
\]

This is a first-order separable linear equation, which has the solution

\[
W(y_1, y_2)(t) = c \exp \left[ - \int p(t) \, dt \right],
\]

where \( c \) is an arbitrary constant. This is summarized in the following theorem.

**Theorem (Abel’s Theorem)** Let \( p(t) \) and \( q(t) \) be continuous on an open interval \( I \), and let \( y_1 \) and \( y_2 \) be solutions of the ODE

\[ L[y] = y'' + p(t)y' + q(t)y = 0. \]

Then the Wronskian \( W(y_1, y_2)(t) \) is given by

\[
W(y_1, y_2)(t) = c \exp \left[ - \int p(t) \, dt \right],
\]

where \( c \) is a constant that depends on \( y_1 \) and \( y_2 \). Furthermore, \( W(y_1, y_2)(t) \) is either never zero in \( I \) (if \( c \) is nonzero) or is zero for all \( t \in I \) (if \( c = 0 \)).

It is interesting to note that except for a constant factor, the Wronskian of two solutions of \( L[y] = 0 \) can be computed even if the solutions \( y_1 \) and \( y_2 \) themselves are unknown.