These notes correspond to Section 3.3 in the text.

**Complex Roots of the Characteristic Equation**

We have learned that a second-order, linear, homogeneous ODE with constant coefficients,

\[ y'' + py' + qy = 0, \]

can be solved by computing the roots of the characteristic equation

\[ \lambda^2 + p\lambda + q = 0. \]

In the case where the roots \( \lambda_1 \) and \( \lambda_2 \) are real and distinct, the functions

\[ y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = e^{\lambda_2 t} \]

form a fundamental set of solutions.

Now, we consider the case where the roots \( \lambda_1 \) and \( \lambda_2 \) are complex, which occurs when the discriminant \( p^2 - 4q < 0 \). Then, we have

\[ \lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta, \quad \alpha = -\frac{p}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2}, \]

where \( i = \sqrt{-1} \). Then, the functions

\[ w_1(t) = e^{\lambda_1 t}, \quad w_2(t) = e^{\lambda_2 t} \]

still form a fundamental set of solutions, as can be verified directly by computing their Wronskian. However, since the solution \( y(t) \) of a particular initial value problem would be real-valued, it is desirable to obtain a fundamental set of solutions whose elements are also real-valued.

To that end, we use Euler’s formula

\[ e^{ix} = \cos x + i \sin x \]

with \( x = \beta t \) and obtain

\[ w_1(t) = e^{\alpha t}(\cos \beta t + i \sin \beta t), \quad w_2(t) = e^{\alpha t}(\cos \beta t - i \sin \beta t). \]

Then, by taking the linear combinations

\[ y_1 = \frac{w_1 + w_2}{2}, \quad y_2 = \frac{w_1 - w_2}{2i}, \]

we obtain real-valued solutions

\[ y_1(t) = e^{\alpha t} \cos \beta t, \quad y_2(t) = e^{\alpha t} \sin \beta t. \]
As the Wronskian of \( y_1 \) and \( y_2 \) is \( \beta e^{2\alpha t} \), these solutions form a fundamental set of solutions, and thus the general solution of the ODE is

\[
y(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants that depend on the initial conditions.

**Example** Consider the initial value problem

\[
y'' + 4y' + 13y = 0, \quad y(0) = -1, \quad y'(0) = 1.
\]

The characteristic equation is

\[
\lambda^2 + 4\lambda + 13 = 0,
\]

which has roots

\[
\lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(13)}}{2} = -2 \pm 3i.
\]

Thus \( \alpha = -2 \) and \( \beta = 3 \), which yields the solutions

\[
y_1(t) = e^{-2t} \cos 3t, \quad y_2(t) = e^{-2t} \sin 3t,
\]

and the general solution

\[
y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t.
\]

Differentiating, we obtain

\[
y'(t) = c_1 (-2e^{-2t} \cos 3t - 3e^{-2t} \sin 3t) + c_2 (-2e^{-2t} \sin 3t + 3e^{-2t} \cos 3t).
\]

Substituting \( t = 0 \) and accounting for the initial conditions yields the equations

\[
y(0) = -1 = c_1, \quad y'(0) = 1 = -2c_1 + 3c_2.
\]

Solving these equations, we obtain the solution

\[
y(t) = -e^{-2t} \cos 3t - \frac{1}{3} e^{-2t} \sin 3t.
\]

Derivation of Euler’s Formula

Using the Maclaurin series for the exponential function,

\[
e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!},
\]

we set \( x = it \), where \( i^2 = -1 \), and split the series into even-numbered and odd-numbered terms to obtain

\[
e^{it} = \sum_{j=0}^{\infty} \frac{i^j t^j}{j!} = \sum_{j=0}^{\infty} \frac{i^{2j} t^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{i^{2j+1} t^{2j+1}}{(2j + 1)!} = \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} + i \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j + 1)!} = \cos t + i \sin t,
\]

where in the last step we have used the Maclaurin series for \( \cos t \) and \( \sin t \).