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Lecture 16 Notes

These notes correspond to Section 14.3 in the text.

Bessel Functions of the Second Kind

When solving the Bessel equation of integer order, Frobenius' method only produces one linearly independent solution. A second solution may be found using reduction of order, but it is not of the same form as a Bessel function of the first kind. Therefore, we refer to it as a *Bessel function of the second kind*, which is also known as a *Neumann function*.

Definition and Series Form

The Neumann function of order ν is defined as follows:

$$Y_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}.$$

This function is clearly a solution of the Bessel equation, as it is a linear combination of solutions. However, if ν is an integer, then $Y_\nu(x)$, as defined, is the indeterminate form $0/0$. Therefore, we need to use l'Hospital's Rule to determine whether the limit as ν approaches an integer n is nonzero, so that we can obtain a meaningful solution.

Applying l'Hospital's Rule yields

$$\begin{aligned} Y_n(x) &= \lim_{\nu\to n} \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \\ &= \lim_{\nu\to n} \frac{-\pi\sin\nu\pi J_\nu(x) + \cos\nu\pi \frac{d}{d\nu}J_\nu(x) - \frac{d}{d\nu}J_{-\nu}(x)}{\pi\cos\nu\pi} \\ &= \lim_{\nu\to n} \frac{1}{\pi}\left[\frac{d}{d\nu}J_\nu(x) - (-1)^n\frac{d}{d\nu}J_{-\nu}(x)\right]. \end{aligned}$$

Using the series definition of $J_\nu(x)$,

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},$$

as well as the *digamma function*

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

we obtain

$$\begin{aligned} Y_n(x) &= \frac{1}{\pi}\left[\sum_{k=0}^\infty \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\ln\left(\frac{x}{2}\right) + (-1)^n\sum_{k=0}^\infty \frac{(-1)^k}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n}\ln\left(\frac{x}{2}\right)\right] - \\ & \frac{1}{\pi}\left[\sum_{k=0}^\infty \frac{(-1)^k\Gamma'(k+n+1)}{k!(k+n)!^2}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=0}^\infty \frac{(-1)^k\Gamma'(k-n+1)}{k!(k-n)!^2}\left(\frac{x}{2}\right)^{2k-n}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[J_n(x)\ln\left(\frac{x}{2}\right) + (-1)^n J_{-n}(x)\ln\left(\frac{x}{2}\right)\right] - \\
&\quad \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k+n+1)}{k!(k+n)!^2}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k-n+1)}{k!(k-n)!^2}\left(\frac{x}{2}\right)^{2k-n}\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \\
&\quad \frac{1}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k+n+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} + (-1)^n\sum_{k=-n}^{\infty}\frac{(-1)^{k+n}\psi(k+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=-n}^{-1}\frac{(-1)^{k+n}\psi(k+1)}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k\psi(k-n+1)}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{(-1)^n}{\pi}\sum_{k=0}^{n-1}\frac{(-1)^k(-1)^{n-k-2}(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right] \\
&= \frac{2}{\pi}J_n(x)\ln\left(\frac{x}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \\
&\quad \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\psi(k+n+1)+\psi(k+1)\right]
\end{aligned}
$$

where we have used the result

$$\lim_{z\to -n}\frac{\psi(z)}{\Gamma(z)} = (-1)^{n-1}n!.$$

We can see from the above expression for $Y_n(x)$ that it is indeed linearly independent of $J_n(x)$, so that we have two linearly independent solutions of the Bessel equation for integer order n. Also, unlike $J_n(x)$, $Y_n(x)$ is singular at $x=0$.

For $n=0$, we have

$$
\begin{aligned}
Y_0(x) &= \frac{2}{\pi}J_0(x)\ln\left(\frac{x}{2}\right) - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left(\frac{x}{2}\right)^{2k}\psi(k+1) \\
&= \frac{2}{\pi}J_0(x)\ln\left(\frac{x}{2}\right) - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left(\frac{x}{2}\right)^{2k}\left[-\gamma+H_k\right] \\
&= \frac{2}{\pi}J_0(x)\left[\gamma+\ln\left(\frac{x}{2}\right)\right] - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left(\frac{x}{2}\right)^{2k}H_k
\end{aligned}
$$

where γ is the Euler-Mascheroni constant. Here we have used the relation

$$\psi(n+1) = -\gamma + \sum_{m=1}^{n} \frac{1}{m}$$

when n is a positive integer.

Integral Representations

Bessel functions of the second kind also have integral representations. We have

$$\begin{aligned}
Y_0(x) &= -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{(t^2-1)^{1/2}}\,dt, \quad x > 0, \\
Y_n(x) &= \frac{1}{\pi}\int_0^\pi \sin(x\sin t - nt)\,dt - \frac{1}{\pi}\int_0^\infty [e^{nt} + (-1)^n e^{-nt}]e^{-x\sinh t}\,dt.
\end{aligned}$$

Recurrence Relations

Bessel functions of the second kind, being solutions of the Bessel equation, satisfy the same recurrence relations as the Bessel functions of the first kind. Specifically,

$$\begin{aligned}
Y_{\nu-1}(x) - Y_{\nu+1}(x) &= 2Y_\nu(x), \\
Y_{\nu-1}(x) + Y_{\nu+1}(x) &= \frac{2\nu}{x}Y_\nu(x).
\end{aligned}$$

We also have the relation

$$Y_{-n}(x) = (-1)^n Y_n(x),$$

when n is an integer.

Wronskian Formulas

If $y_1(x)$ and $y_2(x)$ are solutions of a self-adjoint ODE of the form $p(x)y'' + q(x)y' + r(x)y = 0$, for which $q(x) = p'(x)$, we can use Abel's Theorem to obtain the Wronskian

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = Ce^{-\int \frac{p'(x)}{p(x)}\,dx} = \frac{C}{p(x)},$$

where C is a constant.

By writing the Bessel equation in the form

$$xy'' + y' + (x - \nu^2/x)y = 0,$$

so that it is self-adjoint, we obtain, for non-integer ν,

$$J_\nu(x)J_{-\nu}'(x) - J_\nu'(x)J_{-\nu}(x) = \frac{A_\nu}{x},$$

where A_ν is a constant that depends only on ν, not x.

This constant can be determined by considering any convenient value of x, such as $x = 0$. Examining the leading terms of the series representations of the Bessel functions, which yield approximations for small x,

$$\begin{aligned} J_\nu(x) &\approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \\ J_\nu'(x) &\approx \frac{\nu}{2\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu-1}, \\ J_{-\nu}(x) &\approx \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}, \\ J_{-\nu}'(x) &\approx \frac{-\nu}{2\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}, \end{aligned}$$

we obtain

$$W(J_\nu, J_{-\nu}) = -\frac{2\nu}{x\Gamma(1+\nu)\Gamma(1-\nu)} = -\frac{2\sin\nu\pi}{\pi x}.$$

We conclude that

$$A_\nu = -\frac{2\sin\nu\pi}{x}.$$

When ν is an integer, we obtain $A_\nu = 0$, and therefore the Wronskian is zero. This is expected, since J_n and J_{-n} are linearly dependent when n is an integer.

Using recurrence relations, we obtain the following similar formulas:

$$\begin{aligned} J_\nu J_{-\nu+1} + J_{-\nu}J_{\nu-1} &= \frac{2\sin\nu\pi}{\pi x}, \\ J_\nu J_{-\nu-1} + J_{-\nu}J_{\nu+1} &= -\frac{2\sin\nu\pi}{\pi x}, \\ J_\nu Y_\nu' - J_\nu' Y_\nu &= \frac{2}{\pi x}, \\ J_\nu Y_{\nu+1} - J_{\nu+1}Y_\nu &= -\frac{2}{\pi x}. \end{aligned}$$

Uses of Neumann Functions

Besides completing a set of linearly independent solutions of the Bessel equation, Neumann functions $Y_\nu(x)$ also are useful for physical problems in which there is no requirement of regularity at $x = 0$, such as when modeling electromagnetic waves in coaxial cables, or in quantum mechanical scattering theory.