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MAT 605
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Lecture 20 Notes

These notes correspond to Section 5.3 in the text.

The Linearization and Flow Box Theorems

Recall that a function $f : D \to R$ is *injective* if it is 1-1, meaning that $f(x) \neq f(y)$ if $x \neq y$, and *surjective* if it is onto, meaning that for each $y \in R$ there exists an $x \in D$ such that $f(x) = y$. Finally, f is *bijective* if it is both injective and surjective, and therefore it has an inverse f^{-1}. We now build on these definitions for the purpose of discussing equivalence between systems of DE.

Definition 1 (Homeomorphism) *A function* $\mathbf{f} : D \to R$ *is a* **homeomorphism** *if it is bijective and continuous, and if its inverse* $\mathbf{f}^{-1}$ *is also continuous.*

Note that a homeomorphism is not to be confused with a *homomorphism* from group theory.

Unfortunately, two given systems cannot always be made *differentiably* equivalent, but we can obtain a weaker form of equivalence in most cases.

Definition 2 (Topological Equivalence) *Two systems* $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ *and* $\mathbf{y}' = \mathbf{Y}(\mathbf{y})$ *are* **topologically equivalent** *if there exists a homeomorphism* $\mathbf{f}$ *such that*

1. $\vec{\beta}(t) = \mathbf{f}(\vec{\alpha}(t))$ *is an integral curve of* $\mathbf{Y}$ *for each integral curve* $\vec{\alpha}(t)$ *of* $\mathbf{X}$*, and*
2. $\vec{\alpha}(t) = \mathbf{f}^{-1}(\vec{\beta}(t))$ *is an integral curve of* $\mathbf{X}$ *for each integral curve* $\vec{\beta}(t)$ *of* $\mathbf{Y}$*.*

We can now characterize when two systems are topologically equivalent at fixed points. Recall that a fixed point $\mathbf{c}$ of a system $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ is *hyperbolic* if $A = \mathbf{X}'(\mathbf{c})$ has eigenvalues with all nonzero real parts. That is, A has no eigenvalues that are zero, or purely imaginary.

Theorem 1 (Linearization Theorem) *Let* $\mathbf{c}$ *be a hyperbolic fixed point of the vector field* $\mathbf{X}$*, and let* $A = \mathbf{X}'(\mathbf{c})$*. Then there exists a neighborhood* U *of* $\mathbf{c}$ *and a neighborhood* V *of* $\mathbf{0}$ *such that the restrictions of* $\mathbf{X}$ *to* U *and* $\mathbf{Y}$ *to* V *are topologically equivalent.*

Example 1 Consider the system

$$\begin{aligned} x' &= x^2, \\ y' &= y \end{aligned}$$

This system has a fixed point $(0,0)$, and from

$$\mathbf{X}'(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix}$$

we obtain

$$A = \mathbf{X}'(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\det A = 0$, the fixed point is not hyperbolic, so the systems $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ and $\mathbf{y}' = A\mathbf{y}$ are not topologically equivalent, and the phase portraits are not similar at all. □

Example 2 Consider the system

$$\begin{aligned} x' &= -y - x^3 - xy^2 \\ y' &= x - y^3 - x^2 y \end{aligned}$$

This system has a fixed point $(0,0)$, and from

$$\mathbf{X}'(x,y) = \begin{bmatrix} -3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & -3y^2 - x^2 \end{bmatrix}$$

we obtain

$$A = \mathbf{X}'(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This matrix has purely imaginary eigenvalues $\lambda = \pm i$, so again the fixed point is not hyperbolic. The linear system has a center at the origin, and for this system, the fixed point is stable, but not asymptotically stable. The nonlinear system, on the other hand, has an asymptotically stable fixed point at the origin, that attracts nearby integral curves. □

At a non-fixed point, it can be shown that locally, a nonlinear system is *differentiably* equivalent to a linear system whose integral curves all flow parallel to the x-axis.

Theorem 2 (Flow Box Theorem) *Let $\mathbf{X}$ be a continuously differentiable (C^1) vector field, and suppose $\mathbf{c}$ is not a fixed point of $\mathbf{X}$. Let $\mathbf{Y}(\mathbf{y}) = \mathbf{e}_1 = (1,0,0,\ldots,0)$. Then there exists a neighborhood U of $\mathbf{c}$, a neighborhood $\bar{U}$ of $\mathbf{0}$ such that the restrictions of $\mathbf{X}$ to U and $\mathbf{Y}$ to $\bar{U}$ are differentiably equivalent. That is, there exists a diffeomorphism $\mathbf{F} : U \to \bar{U}$ such that $\mathbf{F}_*(\mathbf{X}) = \mathbf{Y}$.*

The main idea of the proof is to explicitly construct a diffeomorphism $\mathbf{G}$ such that $\mathbf{G}_*(\mathbf{Y}) = \mathbf{X}$. Then, because differentiable equivalence is an equivalence relation, $\mathbf{F} = \mathbf{G}^{-1}$ is the diffeomorphism we seek.

Let $\mathbf{v}_1 = \mathbf{X}(\mathbf{c})$, and let $V = \text{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the orthogonal complement of $\mathbf{v}_1$ in $\mathbb{R}^n$. Let

$$\mathbf{G}(t, a_2, \ldots, a_n) = \vec{\phi}_t(\mathbf{c} + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n).$$

Then if we let $\mathbf{x} = \mathbf{c} + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$, we have

$$\begin{aligned} \mathbf{G}_*(\mathbf{Y})(\mathbf{x}) &= \mathbf{G}'(\mathbf{G}^{-1}(\mathbf{x}))\mathbf{Y}(\mathbf{G}^{-1}(\mathbf{x})) \\ &= \mathbf{G}'(\mathbf{G}^{-1}(\mathbf{x}))\mathbf{e}_1 \\ &= \frac{\partial \mathbf{G}}{\partial t}(0, a_2, \ldots, a_n) \\ &= \frac{\partial}{\partial t}\vec{\phi}_0(\mathbf{c} + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= \mathbf{X}(\vec{\phi}_0(\mathbf{c} + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n)) \\ &= \mathbf{X}(\mathbf{x}). \end{aligned}$$

It can be shown that $\mathbf{G}'(0,\ldots,0) = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, which is invertible, so by the Inverse Function Theorem, $\mathbf{G}$ is invertible in a neighborhood of the origin.

Exercises

Section 5.3: Exercise 3a