These notes correspond to Sections 4.1-4.3 in the text.

The One-Dimensional Wave Equation

Vibrating-String Problem

Newton’s second law applied to an arbitrary segment \([x, x + \Delta x]\) of a vibrating string yields

\[
\Delta x \rho u_{tt} = T[u_x(x + \Delta x, t) - u_x(x, t)] + \Delta x F(x, t) - \Delta x \beta u_t(x, t) - \Delta x \gamma u(x, t),
\]

where \(u(x, t)\) is the displacement of the string from equilibrium, \(\rho\) is the density, \(T\) is the tension, \(F(x, t)\) is the external force, \(-\beta u_t\) is the frictional force against the string, and \(-\gamma u\) is the restoring force. Dividing by \(\Delta x\) and letting \(\Delta x \to 0\) yields the telephone or telegraph equation

\[
u_{tt} = c^2 u_{xx} - \beta u_t - \gamma u + F(x, t),
\]

where \(c^2 = T/\rho\) and \(\beta, \gamma\) and \(F(x, t)\) have been relabeled after dividing by \(\rho\). This equation describes the transverse vibration of the string. It should be noted that because of the second derivative with respect to time, the wave equation has two initial conditions, imposed on \(u(x, 0)\) and \(u_t(x, 0)\), which are the initial position and initial velocity, respectively.

The net force on the segment \([x, x + \Delta x]\) due to the tension is

\[
T \sin \theta_2 - T \sin \theta_1 \approx T[u_x(x + \Delta x, t) - u_x(x, t)],
\]

where \(\theta_1\) and \(\theta_2\) are the angles that the string makes with the \(x\)-axis at \(x\) and \(x + \Delta x\), respectively. The approximations by \(u_x\) follow from right-triangle trigonometry. If the rod has a variable density \(\rho(x)\), then the term of the wave equation arising from the net force due to tension is \((c^2(x)u_x)_x\), rather than \(c^2 u_{xx}\).

Intuitive Interpretation of the Wave Equation

The wave equation states that the acceleration of the string is proportional to the tension in the string, which is given by its concavity.

Applications

Other applications of the one-dimensional wave equation are:

- Modeling the longitudinal and torsional vibration of a rod, or of sound waves. In this case, the coefficient \(c^2\) is called Young’s modulus, which is a measure of the elasticity of the rod.

- Modeling electric current along a wire. This model actually yields the transmission-line equations, which are then manipulated to obtain two wave equations, one for the voltage and one for the current. The coefficients \(c^2\) is inversely proportional to the capacitance and self-inductance per unit length.
The D’Alembert Solution of the Wave Equation

The solution of the Cauchy problem for the wave equation in one space dimension,

\[ u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty, \]

is known as d’Alembert’s solution. We will now derive this solution.

We first introduce the change of variables

\[ \xi = x + ct, \quad \eta = x - ct. \]

Then we have

\[ u_x = u_\xi + u_\eta, \]
\[ u_{xx} = (u_x)_\xi + (u_x)_\eta = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \]
\[ u_t = u_\xi t + u_\eta, \]
\[ u_{tt} = c(u_t)_\xi - c(u_t)_\eta = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}. \]

Substituting these expressions for \( u_{xx} \) and \( u_{tt} \) into the wave equation yields the very simple PDE

\[ u_{\xi\eta} = 0. \]

By integrating with respect to \( \xi \), and then with respect to \( \eta \), we obtain the general solution

\[ u(\xi, \eta) = \Phi(\eta) + \psi(\xi) \]

where the functions \( \Phi(\eta) \) and \( \psi(\xi) \) are chosen so as to satisfy the initial conditions.

Substituting this expression into the initial conditions yields the equations

\[ \Phi(x) + \psi(x) = f(x), \]
\[ -c\Phi'(x) + c\psi'(x) = g(x). \]

This system of equations has the solutions

\[ \Phi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds, \quad \psi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds \]

for some \( x_0 \). It follows that

\[ u(x, t) = \Phi(x - ct) + \psi(x + ct) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]
More on the D’Alembert Solution

Earlier, we learned that the solution of the initial value problem
\[ u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty \]
is given by D’Alembert’s solution
\[ u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]

We now examine how this solution can be interpreted.

The Space-Time Interpretation of D’Alembert’s Solution

First, we consider the case of a zero initial velocity, which has initial conditions
\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad -\infty < x < \infty. \]

Then, the solution is
\[ u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)]. \]

It follows that at any point \((x_0, t_0)\), the solution is equal to the average of the initial displacement \(u(x, 0) = f(x)\) at the two points obtained by backtracking along the lines
\[ x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0 \]
back to the \(x\)-axis.

For example, suppose the initial displacement is given by
\[ f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}. \]

Then, \(u(x, t) = 1/2\) between the lines \(x + ct = \pm 1\), and between the lines \(x - ct = \pm 1\). Where these regions overlap, these values of \(u(x, t)\) are added, and the solution is equal to 1. Outside of these regions, \(u(x, t) = 0\).

Next, we consider the case of a zero initial displacement,
\[ u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty. \]

Then, the solution is
\[ u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]

That is, the solution at \((x_0, t_0)\) is obtained by integrating the initial velocity \(u_t(x, 0) = g(x)\) along the \(x\)-axis from \(x_0 - ct_0\) to \(x_0 + ct_0\).

Therefore, if the initial velocity is given by
\[ g(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}, \]
then \(u(x, t) = (1 + x + ct)/(2c)\) between the lines \(x + ct = \pm 1\), and \(u(x, t) = (1 - x + ct)/(2c)\) between the lines \(x - ct = \pm 1\). Where these regions overlap, the solution is equal to \(t\). Between these two regions, the solution is equal to \(1/c\); everywhere else, it is equal to 0.
Solution of the Semi-Infinite String via the D’Alembert Solution

We now consider a vibrating semi-infinite string with a fixed end, modeled by the IBVP

\[ u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]

\[ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < \infty, \]

\[ u(0,t) = 0, \quad t > 0. \]

As with the infinite string, using the change of variables

\[ \xi = x + ct, \quad \eta = x - ct, \]

we obtain the much simpler PDE

\[ u_{\xi\eta} = 0, \]

which has the general solution

\[ u(x,t) = \phi(\eta) + \psi(\xi) = \phi(x - ct) + \psi(x + ct). \]

As before, we substitute this form of the solution into the initial conditions, and obtain

\[ \phi(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_{x_0}^{x - ct} g(s) \, ds, \quad \psi(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x_0}^{x + ct} g(s) \, ds. \]

However, we can only evaluate \( f(x) \) and \( g(x) \) wherever \( x > 0 \), which presents a problem when \( x - ct < 0 \). To get around this, we apply the boundary condition to the form of \( u(x,t) \) to obtain

\[ u(0,t) = \phi(-ct) + \psi(ct) = 0, \]

or

\[ \phi(-ct) = -\psi(ct). \]

This yields

\[ \phi(x - ct) = -\psi(ct - x) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int_{x_0}^{ct - x} g(s) \, ds, \]

and therefore

\[ u(x,t) = \psi(x + ct) - \psi(ct - x) = \frac{1}{2} [f(x + ct) - f(ct - x)] + \frac{1}{2c} \int_{ct - x}^{x + ct} g(s) \, ds, \quad 0 < x < ct. \]

When \( x \geq ct \), we simply use D’Alembert’s solution as before,

\[ u(x,t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds, \quad x \geq ct. \]

This solution exhibits reflection at the boundary \( x = 0 \).

Exercises

Section 4.6: Exercises 1, 2, 5, 11, 14