These notes correspond to Section 4.4 in the text.

**Domain of Dependence and Region of Influence**

We continue to study the Cauchy problem for the wave equation,

\[ u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \]

From d’Alembert’s solution formula for this problem,

\[ u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds, \]

it can be seen that the solution at any point \((x_0, t_0)\) depends on:

- The values of \(f(x)\) at \(x_0 + ct_0\) and \(x_0 - ct_0\), and
- The values of \(g(x)\) in the interval \(x_0 - ct_0 \leq x \leq x_0 + ct_0\).

The interval \([x_0 - ct_0, x_0 + ct_0]\) forms the base of a characteristic triangle, which is an isosceles triangle that has vertices \((x_0, t_0)\), \((x_0 - ct_0, 0)\) and \((x_0 + ct_0, 0)\). This base \([x_0 - ct_0, x_0 + ct_0]\) is also known as the domain of dependence of \(u\) at \((x_0, t_0)\). It can be seen that if \(f(x)\) and \(g(x)\) vanish within the domain of dependence, then \(u(x_0, t_0) = 0\).

Reversing the concept of domain of dependence, the region of influence of an interval \([a, b]\) consists of those points \((x_0, t_0)\) in the \(xt\)-plane whose domains of dependence overlap with \([a, b]\). It follows that if \((x_0, t_0)\) is outside the region of influence of \([a, b]\), then no initial data within \([a, b]\) can determine \(u(x_0, t_0)\). To determine whether a point \((x, t)\) lies within the region of influence of \([a, b]\), we use its domain of dependence \([x - ct, x + ct]\) and conclude that \((x, t)\) is within the region of influence if and only if

\[ x - ct \leq b, \quad x + ct \geq a. \]

That is, the region of influence forms a truncated characteristic cone bounded by \([a, b]\) on the \(x\)-axis, and the lines \(x + ct = a\) and \(x - ct = b\) for \(t > 0\).

The domain of dependence and region of influence lead to what is called the graphical method for obtaining the solution of the wave equation for given initial data. Recall that the general solution has the form

\[ u(x, t) = \Phi(x - ct) + \Psi(x + ct), \]

where

\[ \Phi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds, \quad \Psi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds \]

where \(x_0\) is a constant. The portion of the solution

\[ \Phi(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_{x_0}^{x-ct} g(s) \, ds \]
is called the forward wave, as it propagates to the right through the evaluation of \( f \) and the antiderivative of \( g \) at \( x - ct \), while the portion
\[
\Psi(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x_0}^{x + ct} g(s) \, ds
\]
is called the backward wave, as it propagates to the left through the evaluation of \( f \) and the antiderivative of \( g \) at \( x + ct \). The graphical method consists of the following steps for evaluating the solution at a particular time \( t \):

1. Graph the forward wave and backward wave at \( t = 0 \).
2. Shift the graph of the forward wave to the right by \( ct \) units, and shift the graph of the backward wave to the left by \( ct \) units.
3. Add the shifted forward and backward waves together.

**Example 1** We consider the Cauchy problem
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0, & -\infty < x < \infty, & t > 0, \\
u(x, 0) &= f(x) = \begin{cases} 2 & |x| \leq a, \\ 0 & |x| > a, \end{cases} & -\infty < x < \infty, \\
u_t(x, 0) &= g(x) = 0, & -\infty < x < \infty.
\end{align*}
\]
The forward wave is
\[
\Phi(x - ct) = \begin{cases} 1 & |x - ct| \leq a, \\ 0 & |x - ct| > a, \end{cases}
\]
while the backward wave is
\[
\Psi(x + ct) = \begin{cases} 1 & |x + ct| \leq a, \\ 0 & |x + ct| > a. \end{cases}
\]

\( \square \)

**Example 2** We now consider the Cauchy problem
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, & -\infty < x < \infty, & t > 0, \\
u(x, 0) &= f(x) = 0, & -\infty < x < \infty, \\
u_t(x, 0) &= g(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}
\end{align*}
\]
The forward wave is
\[
-\frac{1}{2} \int_0^{x-t} g(s) \, ds = -\frac{\max\{0, x-t\}}{2},
\]
while the backward wave is
\[
\frac{1}{2} \int_0^{x+t} g(s) \, ds = \frac{\max\{0, x+t\}}{2}.
\]
The characteristic lines \( x + t = 0 \) and \( x - t = 0 \) can be used to graph the shifted backward and forward waves, respectively. Because the forward wave is zero except for \( x > t \), the solution is
\[
u(x, t) = \begin{cases} 0 & x < -t, \\ \frac{1}{2} (x+t) & -t \leq x \leq t, \\ t & x > t, \end{cases}
\]
as the \( x \)-dependent terms in the forward and backward waves cancel each other out for \( x > t \). \( \square \)