Nonhomogeneous Boundary Conditions

In order to use separation of variables to solve an IBVP, it is essential that the boundary conditions (BCs) be homogeneous. If they are not, then it is possible to transform the IBVP into an equivalent problem in which the BCs are homogeneous. We illustrate this process with some examples.

Transforming Time-Independent BCs

Consider the IBVP

\[ u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \]

\[ u(0, t) = k_1, \quad u(L, t) = k_2, \quad t > 0, \]

\[ u(x, 0) = \phi(x), \quad 0 < x < L. \]

We express the solution \( u(x, t) \) in the form

\[ u(x, t) = v(x, t) + w(x), \]

where \( w(x) \) satisfies the boundary conditions, and therefore \( v(x, t) \) satisfies boundary conditions of the same form as those for \( u(x, t) \), except that they are homogeneous. The function \( w \) depends only on \( x \), not \( t \), because the boundary values \( k_1 \) and \( k_2 \) do not depend on \( t \) either.

We also require that \( v(x, t) \) satisfies the same PDE as \( u(x, t) \). Because the PDE is linear, it follows that \( w(x) \) must be a solution of the PDE as well. However, because \( w \) is independent of \( t \), \( w_t = 0 \), and therefore we must also have \( w_{xx} = 0 \), which means \( w(x) \) must be a linear function of \( x \). In order to meet this requirement, and the requirement that \( w(x) \) satisfy the boundary conditions, we can use Lagrange interpolation to obtain

\[ w(x) = \frac{k_1}{0 - L} x - \frac{L}{0 - L} + \frac{k_2}{L - 0} x - \frac{0}{L - 0} = \frac{x}{L} (k_2 - k_1). \]

We conclude that \( v(x, t) \) must be a solution of the IBVP

\[ v_t = \alpha^2 v_{xx}, \quad 0 < x < L, \quad t > 0, \]

\[ v(0, t) = 0, \quad v(L, t) = 0, \quad t > 0, \]

\[ v(x, 0) = \phi(x) - w(x), \quad 0 < x < L. \]

This IBVP can be solved using separation of variables. Then, the solution \( v(x, t) \) can be added to \( w(x) \) to obtain the solution \( u(x, t) \) of the original IBVP.
Transforming Time-Dependent BCs

We now consider an IBVP in which the boundary conditions involve functions that vary over time:

\[ u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \]
\[ u(0, t) = a(t), \quad u_x(L, t) + hu(L, t) = b(t), \quad t > 0, \]
\[ u(x, 0) = \phi(x), \quad 0 < x < L. \]

Proceeding as before, we express the solution \( u(x, t) \) in the form

\[ u(x, t) = v(x, t) + w(x, t), \]

where \( w(x, t) \) satisfies the boundary conditions, and therefore \( v(x, t) \) satisfies boundary conditions of the same form as those for \( u(x, t) \), except that they are homogeneous. Unlike the previous example, the function \( w \) depends on both \( x \) and \( t \), because the boundary values \( a(t) \) and \( b(t) \) depends on \( t \).

We would like \( v(x, t) \) to satisfy a PDE of the same form as the one satisfied by \( u(x, t) \), but it will not be exactly the same PDE in this case. As in the previous example, we can construct a linear function of \( x \) that will satisfy the boundary conditions, but it will also depend on \( t \). Using Lagrange interpolation as before, we prescribe that \( w(x, t) \) has the form

\[ w(x, t) = a(t) \frac{x - L}{0 - L} + k_2(t) \frac{x - 0}{L - 0} = a(t) + \frac{x}{L} [k_2(t) - a(t)], \]

where the function \( k_2(t) \) is to be determined. Substituting this form for \( w(x, t) \) into the BC at \( x = L \) yields

\[ w_x(L, t) + hw(L, t) = \frac{1}{L} [k_2(t) - a(t)] + ha(t) + h[k_2(t) - a(t)] = b(t), \]

which yields

\[ k_2(t) = a(t) + \frac{L}{1 + Lh} [b(t) - ha(t)], \]

and therefore

\[ w(x, t) = a(t) \frac{L - x}{L} + \frac{x}{1 + Lh} [b(t) - ha(t)]. \]

Because \( w(x, t) \) is a linear function of \( x \), we have \( w_{xx} = 0 \). By substituting the above form of \( u(x, t) \) into the original IBVP, we obtain the new IBVP

\[ v_t = \alpha^2 v_{xx} - w_t, \quad 0 < x < L, \quad t > 0, \]
\[ v(0, t) = 0, \quad v_x(L, t) + hv(L, t) = 0, \quad t > 0, \]
\[ v(x, 0) = \phi(x) - w(x, 0), \quad 0 < x < L. \]

The resulting PDE is no longer homogeneous, so we cannot apply separation of variables directly. Later, we will learn how to handle this kind of inhomogeneity.

Example 1
The General Case

To homogenize the general boundary conditions
\[
\alpha u(0, t) + \beta u_x(0, t) = a(t), \\
\gamma u(L, t) + \delta u_x(L, t) = b(t),
\]
we need a function \(w(x, t)\) that is linear in \(x\), and satisfies these boundary conditions. Using Lagrange interpolation, we obtain
\[
w(x, t) = k_1(t)\frac{L-x}{L} + k_2(t)\frac{x}{L},
\]
where \(k_1(t)\) and \(k_2(t)\) are to be determined. This form of \(w(x, t)\) ensures that
\[
w(0, t) = k_1(t), \quad w(L, t) = k_2(t).
\]

We also have
\[
w_x(x, t) = \frac{1}{L}[k_2(t) - k_1(t)].
\]

Substituting \(w(x, t)\) into the boundary conditions for \(u(x, t)\) yields
\[
\alpha k_1(t) + \frac{\beta}{L}[k_2(t) - k_1(t)] = a(t), \\
\gamma k_2(t) + \frac{\delta}{L}[k_2(t) - k_1(t)] = b(t).
\]

Rearranging, we obtain
\[
\left(\alpha - \frac{\beta}{L}\right)k_1(t) + \frac{\beta}{L}k_2(t) = a(t), \\
-\frac{\delta}{L}k_1(t) + \left(\gamma + \frac{\delta}{L}\right)k_2(t) = b(t).
\]

In matrix-vector form, this system of linear equations can be written as
\[
\begin{bmatrix}
\alpha L - \beta & \beta \\
-\delta & \gamma L + \delta
\end{bmatrix}
\begin{bmatrix}
k_1(t) \\
k_2(t)
\end{bmatrix}
= 
\begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix}.
\]

Using the formula for the inverse of a \(2 \times 2\) matrix,
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad-bc}
\begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix},
\]
we obtain the solutions
\[
\begin{bmatrix}
k_1(t) \\
k_2(t)
\end{bmatrix}
= 
\begin{bmatrix}
\alpha L - \beta & \beta \\
-\delta & \gamma L + \delta
\end{bmatrix}^{-1}
\begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix}
= 
\frac{1}{\alpha \gamma L + (\alpha \delta - \gamma \beta)}
\begin{bmatrix}
(\gamma L + \delta)a(t) - \beta b(t) \\
\delta a(t) + (\alpha L - \beta)b(t)
\end{bmatrix}.
\]

We can see from these solution formulas that not all boundary conditions can be homogenized in this way, because the solutions cannot be found if the determinant of the coefficient matrix, \(\alpha \gamma L^2 + L(\alpha \delta - \gamma \beta)\), is zero.
Example 2 Consider Neumann boundary conditions

\[ u_x(0, t) = a(t), \quad u_x(L, t) = b(t). \]

These boundary conditions cannot be homogenized using the approach described above, because with \( \alpha = \gamma = 0 \), the determinant of the coefficient matrix is zero. Instead, we let \( f = u_x \) and consider the Dirichlet boundary conditions

\[ f(0, t) = a(t), \quad f(L, t) = b(t). \]

Then, following the approach of the preceding discussion, we express \( f(x, t) \) in the form

\[ f(x, t) = g(x, t) + h(x, t) \]

where \( g(x, t) \) satisfies an IBVP with homogeneous boundary conditions, and

\[ h(x, t) = a(t) \frac{L - x}{L} + b(t) \frac{x}{L}. \]

Then, we can express \( u(x, t) \) in the form

\[ u(x, t) = v(x, t) + w(x, t) \]

where \( v(x, t) \) satisfies an IBVP with homogeneous boundary conditions, and

\[ w(x, t) = \int h(x, t) \, dx = a(t) \frac{Lx - \frac{1}{2}x^2}{L} + b(t) \frac{x^2}{2L} = [b(t) - a(t)] \frac{x^2}{2L} + a(t)x. \]

In determining the PDE for \( v(x, t) \), it must be taken into account that \( w_{xx} = [b(t) - a(t)]/L \), since it is no longer a linear function of \( x \). \( \square \)