Applications of the Maximum Principle

The maximum principle for Laplace’s equation is very useful for proving uniqueness of solutions of various problems.

**Theorem 1** The Dirichlet problem in a bounded domain $D$ has at most one solution in $C^2(D) \cap C(\overline{D})$.

**Proof:** Assume that $u_1$ and $u_2$ are solutions of the Dirichlet problem on $D$. Then the difference $v \equiv u_1 - u_2$ is also a solution of the Dirichlet problem, with $v \equiv 0$ on $\partial D$. By the weak maximum principle, the maximum and minimum of $v$ are assumed on the boundary, so $v \equiv 0$ in $D$, and therefore $u_1 = u_2$. □

Another application is stability of solutions.

**Theorem 2** Let $u_1, u_2 \in C^2(D) \cap C(\overline{D})$ be solutions of $\Delta u = F$ in a bounded domain $D$ with Dirichlet boundary conditions $u_1 = g_1, u_2 = g_2$ on $\partial D$. Then

$$\max_D |u_1(x,y) - u_2(x,y)| \leq \max_{\partial D} |g_1(x,y) - g_2(x,y)|.$$

**Proof:** Let $v \equiv u_1 - u_2$. Then $v$ is a harmonic function with Dirichlet boundary data $g \equiv g_1 - g_2$. By the weak maximum principle,

$$\min_{\partial D} g(x,y) \leq v(x,y) \leq \max_{\partial D} g(x,y), \quad (x,y) \in D,$

from which the theorem follows. □

**Green’s Identities**

The following fundamental theorem from vector calculus can be used to prove several theorems regarding uniqueness of solutions.

**Theorem 3 (Divergence Theorem)** Let $D$ be a bounded domain with piecewise smooth boundary $\partial D$, and let $\psi \in C^1(D) \cap C(\overline{D})$. Then

$$\int_D \nabla \cdot \psi(x,y) \, dx \, dy = \int_{\partial D} \psi(x(s), y(s)) \cdot n \, ds.$$

From this theorem, we obtain the three Green’s identities:
• From \( \psi = \nabla u \), we obtain
  \[
  \int_D \Delta u \, dx \, dy = \int_{\partial D} \partial_n u \, ds.
  \]

• From \( \psi = v\nabla u - u\nabla v \) and the Product Rule for differentiation, we obtain
  \[
  \int_{\partial D} v \partial_n u - u \partial_n v \, ds = \int_D \nabla \cdot (v\nabla u - u\nabla v) \, dx \, dy
  = \int_D v\Delta u + \nabla u \cdot \nabla v - u\Delta v - \nabla u \cdot \nabla v \, dx \, dy
  = \int_D v\Delta u - u\Delta v \, dx \, dy.
  \]

• From \( \psi = v\nabla u \) we obtain
  \[
  \int_{\partial D} v \partial_n u \, ds = \int_D v\Delta u + \nabla u \cdot \nabla v \, dx \, dy.
  \]

We can now prove the following uniqueness results.

**Theorem 4** Let \( D \) be a bounded domain with smooth boundary \( \partial D \).

1. The Dirichlet problem
   \[
   \Delta u = F \text{ on } D, \quad u = g \text{ on } \partial D
   \]
   has at most one solution.

2. If \( \alpha \geq 0 \), the Robin problem
   \[
   \Delta u = F \text{ on } D, \quad u + \alpha \partial_n u = g \text{ on } \partial D
   \]
   has at most one solution.

3. Any two solutions of the Neumann problem
   \[
   \Delta u = f \text{ on } D, \quad \partial_n u = g \text{ on } \partial D
   \]
   differ by a constant.

**Proof:** For each of these three problems, let \( u_1, u_2 \) be two solutions and let \( v \equiv u_1 - u_2 \). Then \( v \) is harmonic in \( D \), and satisfies a homogeneous boundary condition.

1. By the weak maximum principle, \( v = 0 \) on \( \partial D \).

2. From the third Green's identity, and the fact that \( \Delta v = 0 \) in \( D \), we have
   \[
   \int_{\partial D} v \partial_n v \, ds = \int_D \nabla v \cdot \nabla v \, dx \, dy.
   \]
   From the boundary condition, \( v = -\alpha \partial_n v \) on \( \partial D \), which yields
   \[
   - \int_{\partial D} \alpha (\partial_n v)^2 \, ds = \int_D \|\nabla v\|^2 \, dx \, dy.
   \]
Because the integral on the left side is nonpositive, and the integral on the right side is nonnegative, both integrals must be zero. Because neither integrand changes sign, it follows that both integrands must be identically zero. Therefore, $\alpha \partial_{n}v \equiv 0$ on $\partial D$, but from the boundary condition, we then have $v \equiv 0$ on $\partial D$. By the weak maximum principle, $v \equiv 0$ in $D$.

3. From the third Green’s identity, and the fact that $\partial_{n}v = 0$ on $\partial D$ and $\Delta v = 0$ in $D$, we have

$$\int_{D} \nabla v \cdot \nabla v \, dx \, dy = \int_{D} \| \nabla v \|^{2} \, dx \, dy = 0.$$ 

It follows that $\nabla v \equiv 0$ in $D$, so $v$ must be a constant.

The Maximum Principle for the Heat Equation

Consider the following IBVP for heat equation on a bounded three-dimensional domain, that is also bounded in time:

$$u_t = k \Delta u, \quad (x, y, z) \in D, \quad t > 0,$$

$$Q_T = \{(x, y, z, t) | (x, y, z) \in D, 0 < t \leq T \}$$

$$\partial_{P}Q_T = \{ D \times \{0\} \} \cup \{ \partial D \times [0, T] \}$$

Let $C_H$ be the class of functions that are twice differentiable in $Q_T$ with respect to $x, y, z$, once differentiable with respect to $t$, and continuous in $Q_T$. We then have the following preliminary result.

**Proposition 5** Let $v \in C_H$ satisfy $v_t - k \Delta v < 0$ in $Q_T$. Then $v$ has no local maximum in $Q_T$, and $v$ achieves its maximum on $\partial_{P}Q_T$.

**Proof:** Let $v$ have a local maximum at $q \in Q_T$. Then $v_t(q) = 0$, and $\Delta v(q) < 0$, which contradicts the assumption $v_t - k \Delta v < 0$ in $Q_T$. Because $v$ is continuous on $Q_T$, $v$ has a maximum on $\overline{Q_T}$, which must be assumed on the boundary $\partial Q_T$. If this maximum is achieved at a point $q \in \{ D \times \{T\} \}$, we must have $v_t(q) \geq 0$, and because it is a local maximum in $(x, y, z)$, $\Delta v(q) < 0$. By continuity, this again contradicts the assumption $v_t - k \Delta v < 0$ in $Q_T$. □

The preceding proposition leads us to the following maximum principle for the heat equation.

**Theorem 6** Let $u \in C_H$ be a solution of the heat equation (1) in $Q_T$. Then $u$ achieves its maximum and minimum values on $\partial_{P}Q_T$.

**Proof:** Let $v = u - \epsilon t$. Then $v_t - k \Delta v = u_t - k \Delta u - \epsilon$, and therefore $v_t - k \Delta v < 0$. By the preceding proposition, $v$ has no local maximum in $Q_T$, and therefore the maximum of $u$ is assumed on $\partial_{P}Q_T$. Furthermore,

$$\max_{Q_T} v = \max_{\partial_{P}Q_T} v \leq M = \max_{\partial_{P}Q_T} u,$$

because $v \leq u$ in $\overline{Q_T}$. By rearranging $v = u - \epsilon t \leq M$, we obtain $u \leq M + \epsilon T$ on $Q_T$. Letting $\epsilon \to 0$, we obtain $u \leq M$ on $Q_T$, and therefore the maximum of $u$ is assumed on $\partial_{P}Q_T$, since $M$ is defined to be the maximum of $u$ on $\partial_{P}Q_T$. □

The preceding theorem has this consequence.
**Theorem 7** Let $u_1, u_2$ be solutions of the heat equation
\[ u_t - k\Delta u = F(x, y, z, t), \quad (x, y, z) \in D, \quad 0 < t < T, \]
with initial conditions

\[ u_1(x, y, z, 0) = f_1(x, y, z), \quad u_2(x, y, z, 0) = f_2(x, y, z), \quad (x, y, z) \in D; \]

and boundary conditions

\[ u_1(x, y, z, t) = h_1(x, y, z, t), \quad u_2(x, y, z, t) = h_2(x, y, z, t), \quad (x, y, z) \in \partial D, \quad 0 < t < T. \]

Then
\[ |u_1 - u_2| \leq \max_{D} |f_2 - f_1| + \max_{\partial D \times \{t > 0\}} |h_1 - h_2|. \]

**Proof:** Let $v \equiv u_1 - u_2$. Then $v$ solves the IBVP
\[ v_t - k\Delta u = 0, \quad (x, y, z) \in D, \quad 0 < t < T, \]
\[ v(x, y, z, 0) = f_1(x, y, z) - f_2(x, y, z), \quad (x, y, z) \in D, \]
\[ v(x, y, z, t) = h_1(x, y, z, t) - h_2(x, y, z, t), \quad (x, y, z) \in \partial D, \quad 0 < t < T. \]

It follows from the preceding theorem that $v$ assumes its maximum and minimum on $\partial PQT$. \(\square\)

This theorem states that the solution of the heat equation is stable; by setting $f_1 = f_2$ and $h_1 = h_2$, we obtain uniqueness as well. Specializing to the 1-D heat equation, we obtain the following result.

**Corollary 8** Let
\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2t} \quad (2) \]
be the solution of the IBVP
\[ u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \]
\[ u(0, t) = u(L, t) = 0, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad 0 < x < L. \]

If the series
\[ f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \]
converges uniformly on $(0, L)$, then the series (2) converges uniformly on $(0, L) \times (0, T)$, and $u$ is a classical solution.

**Proof:** From the uniform convergence of the series for $f(x)$, we have that for any $\epsilon > 0$ there exists $N_{\epsilon}$ such that for $k, l \geq N_{\epsilon}$,
\[ \left| \sum_{n=k}^{l} B_n \sin \frac{n\pi x}{L} \right| < \epsilon, \quad 0 < x < L. \]
Because
\[
\sum_{n=k}^{l} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}
\]
is a classical solution of the heat equation, the maximum principle applies, and therefore this solution is also bounded by \( \epsilon \) on \((0, L) \times (0, T)\). It follows that the series (2) converges uniformly on \((0, L) \times (0, T)\). Because \( u \) satisfies the initial condition, boundary conditions, and PDE, it is a classical solution. \( \square \)

We conclude with a maximum principle for the Neumann problem.

**Proposition 9** Let \( u(x,t) \) be the solution of the IBVP
\[
\begin{align*}
    u_t &= ku_{xx}, \quad a < x < b, \quad t > 0, \\
    u(x,0) &= f(x), \quad a < x < b, \\
    u_x(a,t) &= 0, \quad u_x(b,t) = 0, \quad 0 < t < T.
\end{align*}
\]
Let \( f \in C^1([a,b]) \) satisfy \( f'(a) = f'(b) = 0 \), and let \( Q_T = (a,b) \times (0,T) \). Then
\[
\max_{Q_T} |u_x| \leq \max_{(a,b)} |f'(x)|.
\]

**Proof:** Let \( w = u_x \). Then \( w \) satisfies the heat equation with Dirichlet boundary conditions, with initial condition \( w(x,0) = f'(x) \). The proposition then follows from the maximum principle for the heat equation. \( \square \)

**Exercises**
Section 7.9: Exercises 1, 4, 5, 8