These notes correspond to Sections 7.7-7.8 in the text.

**Separation of Variables for Elliptic Problems**

We now use separation of variables to solve an elliptic problem on a rectangle,

\[ \Delta u = 0, \quad 0 < x < L, \quad 0 < y < M, \]

\[ u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = a(x), \quad u(x, M) = b(x). \]

To apply separation of variables, we assume the solution has the form

\[ u(x, y) = X(x)Y(y). \]

Substituting this form into the PDE yields

\[ X''Y + XY'' = 0, \]

or, after rearranging,

\[ \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \]

This yields the ODEs

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0, \]

\[ Y'' - \lambda Y = 0. \]

These ODEs can easily be solved to obtain

\[ X_n(x) = \sin \frac{n\pi x}{L}, \quad Y_n(y) = A_n \sinh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi (M - y)}{L}. \]

Our solution then has the form

\[ u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[ A_n \sinh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi (M - y)}{L} \right]. \quad (1) \]

Applying the remaining boundary conditions yields the equations

\[ u(x, 0) = a(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi M}{L}, \quad \text{(2)} \]

\[ u(x, M) = b(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi M}{L}. \quad \text{(3)} \]

Taking the inner product of both sides of each equation with each eigenfunction yields

\[ A_n = \frac{2}{L \sinh \frac{n\pi M}{L}} \int_{0}^{L} b(x) \sin \frac{n\pi x}{L} \, dx, \quad B_n = \frac{2}{L \sinh \frac{n\pi M}{L}} \int_{0}^{L} a(x) \sin \frac{n\pi x}{L} \, dx. \]

If the series (2), (3) converge uniformly to \( a(x) \) and \( b(x) \), respectively, then (1) converges uniformly to a classical solution of the BVP.

This procedure can be adapted to several variations of this problem:
• If the boundary conditions at \( y = 0 \) and \( y = M \) are homogeneous, rather than those at \( x = 0 \) and \( x = L \), then the same solution procedure can be followed, except the roles of \( x \) and \( y \) are reversed.

• If the PDE is not homogeneous (that is, if Poisson’s equation is to be solved), then the right-side function \( F(x, y) \) can be expanded in the basis of eigenfunctions, as with time-dependent nonhomogeneous PDEs.

• If the boundary conditions are not homogeneous in either direction, then we can let \( u(x, y) = v(x, y) + w(x, y) \), where \( w(x, y) \) is a linear or quadratic function that satisfies the boundary conditions, and \( v(x, y) \) is the solution of \( \Delta v = F(x, y) - \Delta w \), where \( \Delta u = F(x, y) \) is the original PDE.

The Interior Dirichlet Problem for a Circle

We consider the boundary value problem (BVP) on a circle of radius \( \hat{R} \),

\[
\nabla^2 u = 0, \quad x^2 + y^2 < \hat{R}^2,
\]

which a Dirichlet boundary condition

\[
u = g, \quad x^2 + y^2 = \hat{R}^2.
\]

It is more convenient to work in polar coordinates,

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < \hat{R}, \quad 0 \leq \theta < 2\pi,
\]

\[
u(\hat{R}, \theta) = g(\theta), \quad 0 \leq \theta < 2\pi.
\]

Separation of Variables

We first solve this problem using separation of variables. We assume the solution has the form

\[
u(r, \theta) = R(r)\Theta(\theta).
\]

Substituting this form of the solution into the PDE yields

\[
R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} \Theta'' = 0,
\]

which can be rearranged to obtain

\[
-\frac{r^2 R'' + r R'}{R} = \frac{\Theta''}{\Theta} = k,
\]

where \( k \) is the separation constant. We then obtain the ODEs

\[
\Theta'' - k\Theta = 0,
\]

\[
r^2 R'' + r R' + kR = 0.
\]

We consider the three possibilities for \( k \):
1. $k = 0$: Then $\Theta = A + B\theta$, where $A$ and $B$ are constants. However, $\Theta$ must be a $2\pi$-periodic function, which is not possible unless $B = 0$.

2. $k > 0$: Then $\Theta = Ae^{k\theta} + Be^{-k\theta}$, where $A$ and $B$ are constants. This function cannot be $2\pi$-periodic unless $A = B = 0$, so this scenario is not considered.

3. $k < 0$: Then we let $k = -\lambda^2$, in which case
   \[ \Theta = A\cos(\lambda\theta) + B\sin(\lambda\theta) \]
   is the general solution. In order to be $2\pi$-periodic, we must require that $\lambda$ be a positive integer.

We conclude that the ODE for $\Theta$ has the solutions
   \[ \Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0, 1, 2, \ldots, \]
where $A_n$ and $B_n$ are constants.

We then consider the ODE for $R$,
   \[ r^2R'' + rR' - n^2R = 0, \]
which is an Euler equation. Using the substitution $z = \ln x$, we can transform this equation into a second-order linear ODE with constant coefficients, which yields the general solution
   \[ R(r) = Ar^n + Br^{-n}, \]
where $A$ and $B$ are constants. In order for the solution to be well-defined at the center of the circle, we set $B = 0$. It follows that a solution of Laplace’s equation on the circle is
   \[ u_n(r, \theta) = \left( \frac{r}{\tilde{R}} \right)^n [A_n\cos(n\theta) + B_n\sin(n\theta)], \quad n = 0, 1, 2, \ldots, \]
where we have added the constant factor $1/\tilde{R}^n$ for convenience. Thus we have the general solution of our BVP,
   \[ u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{\tilde{R}} \right)^n [A_n\cos(n\theta) + B_n\sin(n\theta)], \]
where again we have added the constant factor of $1/2$ in the first term for convenience.

We must choose the constants $A_n$ and $B_n$ to satisfy the boundary condition. Substituting $r = \tilde{R}$ yields the equation
   \[ g(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta). \]
It follows that these constants are the coefficients of the Fourier series of $g(\theta)$,
   \[ A_n = \frac{1}{\pi} \int_{0}^{2\pi} g(\alpha) \cos(n\alpha) \, d\alpha, \quad n = 0, 1, 2, \ldots, \]
   \[ B_n = \frac{1}{\pi} \int_{0}^{2\pi} g(\alpha) \sin(n\alpha) \, d\alpha, \quad n = 1, 2, \ldots. \]
This completes the solution of the BVP.
The Poisson Integral Formula

Using the formulas for the constants $A_n$ and $B_n$ in the above formula for the solution, we can obtain an alternative formula. Using $R$ in place of $\tilde{R}$ (not to be confused with the function $R(r)$ from separation of variables), we have

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \left[ \cos(n\theta) \cos(n\alpha) + \sin(n\theta) \sin(n\alpha) \right] \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos(n(\theta - \alpha)) \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \left( e^{in(\theta - \alpha)} + e^{-in(\theta - \alpha)} \right) \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{r}{R} e^{i(\theta - \alpha)} \right)^n + \sum_{n=1}^{\infty} \left( \frac{r}{R} e^{-i(\theta - \alpha)} \right)^n \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ 1 + \frac{re^{i(\theta - \alpha)}}{R - re^{i(\theta - \alpha)}} + \frac{re^{-i(\theta - \alpha)}}{R - re^{-i(\theta - \alpha)}} \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ 1 + \frac{re^{i(\theta - \alpha)}(R - re^{-i(\theta - \alpha)}) + re^{-i(\theta - \alpha)}(R - re^{i(\theta - \alpha)})}{R^2 - 2rR \cos(\theta - \alpha) + r^2} \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ 1 + \frac{2rR \cos(\theta - \alpha) - 2r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2} \right] g(\alpha) \, d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \alpha) + r^2} \right] g(\alpha) \, d\alpha.$$

This formula is called the Poisson integral formula. It is interesting to note that from this formula, it can easily be seen that

$$u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) \, d\alpha.$$  

That is, the value of the solution at the center of the circle is equal to the average value of $g$ on the boundary. Furthermore, by the Law of Cosines, the denominator in the integrand is the length of the side opposite the angle $|\theta - \alpha|$ of the triangle with vertices $(0, 0)$, $(r, \theta)$, and $(R, \alpha)$.

For a BVP defined on a non-circular domain, this formula can be applied by first conformally mapping the domain to a circle, using the formula on the transformed problem, and then transforming back to the original domain.