These notes correspond to Sections 2.1-2.3 in the text.

The Method of Characteristics

Consider the general first-order linear initial value problem

\[ a(x,t)u_x + b(x,t)u_t = c(x,t)u, \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x,0) = f(x), \quad -\infty < x < \infty. \]

To solve this problem, we will change independent variables \((x,t)\) to new variables \((\tau,s)\) in order to obtain a new PDE that is easier to solve. To define these new variables, we solve the IVPs

\[ \frac{dx}{d\tau} = a(x,t), \quad x(0,s) = s, \]

\[ \frac{dt}{d\tau} = b(x,t), \quad t(0,s) = 0. \]

Then, by the Chain Rule, the PDE reduces to the ODE

\[ a(x,t)u_x + b(x,t)u_t = u_x x_{\tau} + u_t t_{\tau} = \frac{du}{d\tau} = c(x(\tau,s),y(\tau,s))u, \]

with initial condition

\[ u(0,s) = f(s). \]

Once we solve this problem, we change back to the original variables \((x,t)\) to obtain the solution.

This approach is known as the method of characteristics. The new variables \((\tau,s)\) define characteristic curves \{\(x(\tau,s),t(\tau,s),u(\tau,s)\}\}, each of which has the initial point \((0,s)\). The initial data propagates along these curves. The \(\tau\)-coordinate indicates points along a characteristic curve, whereas the \(s\)-coordinate indicates the initial point \((t = 0)\) for a given curve. For each \(s\), the initial value \(f(s)\) is evolved along the curve starting at the point \((x,t) = (0,s)\) according to the ODE obtained by expressing the PDE in characteristic variables.

For the purpose of understanding the behavior of the solution, it is often helpful to examine the projections of the characteristic curves onto the \((x,t)\)-plane. For this reason, we refer to the plane curves \{\(x(\tau,s),t(\tau,s)\}\} as characteristics. For a linear PDE, the coefficients \(a\) and \(b\) do not depend on \(u\), so it is always possible to solve for the characteristics independently of the solution \(u\), but for nonlinear PDE, this is not the case.

Example 1 We will solve the IVP

\[ u_x + u_t + 2u = 0, \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x,0) = \sin x, \quad -\infty < x < \infty. \]

The characteristics satisfy the equations

\[ \frac{dx}{d\tau} = 1, \quad x(0,s) = s, \]
\[ \frac{dt}{d\tau} = 1, \quad t(0, s) = 0. \]

These equations have the solutions
\[ x = \tau + s, \quad t = \tau. \]

The PDE thus reduces to the ODE
\[ \frac{du}{d\tau} + 2u = 0, \quad u(0, s) = \sin s, \]
which has the solution
\[ u(\tau, s) = e^{-2\tau} \sin s. \]

Changing back to the original variables yields
\[ u(x, t) = e^{-2t} \sin(x - t). \]

\[ \Box \]

**Example 2** We will solve the IVP
\[ xu_x + u_t + tu = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = F(x). \]

The characteristics satisfy the equations
\[ \frac{dx}{d\tau} = x, \quad x(0, s) = s, \]
\[ \frac{dt}{d\tau} = 1, \quad t(0, s) = 0. \]

These equations have the solutions
\[ x = se^\tau, \quad t = \tau. \]

The PDE thus reduces to the ODE
\[ \frac{du}{d\tau} + \tau u = 0, \quad u(0, s) = F(s), \]
which has the solution
\[ u(\tau, s) = e^{-\tau^2/2} F(s). \]

Changing back to the original variables yields
\[ u(x, t) = e^{-t^2/2} F(xe^{-t}). \]

\[ \Box \]
Nonlinear First-Order Equations

We consider the problem of modeling traffic flow. Let $u(x, t)$ represent the density of cars at the point $x$ of a highway at time $t$. Then $u$ satisfies the following conservation equation: on any segment $[a, b]$ of the highway,

$$\frac{d}{dt} \int_a^b u(x, t) \, dx = f(a, t) - f(b, t),$$

where $f(x, t)$ is the flux, which is the cars per minute passing the point $x$. Both sides of the equation represent the change in the number of cars within $[a, b]$. Using the Fundamental Theorem of Calculus, we obtain

$$\int_a^b u_t(x, t) \, dx = -\int_a^b f_x(x, t) \, dt$$

which is rearranged to

$$\int_a^b u_t(x, t) + f_x(x, t) \, dx = 0.$$

Because the segment $[a, b]$ is arbitrary, we conclude that $u$ is a solution of the PDE

$$u_t + f_x = 0.$$

We now attempt to solve the conservation equation

$$u_t + f_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = \phi(x), \quad -\infty < x < \infty.$$

Using the Chain Rule, we rewrite the PDE as

$$u_t + \frac{df}{du} u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

or

$$u_t + g(u) u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

where $g(u) = df/du$.

The characteristics satisfy the equations

$$\frac{dx}{d\tau} = g(u(0, s)), \quad x(0, s) = s,$$

$$\frac{dt}{d\tau} = 1, \quad t(0, s) = 0$$

and the IVP reduces to

$$\frac{du}{d\tau} = 0, \quad u(0, s) = \phi(s).$$

Since the solution is independent of $\tau$, the characteristic curves are defined by

$$x = s + g(\phi(s))\tau, \quad t = \tau.$$

The reduced IVP for $u$ has the solution

$$u(\tau, s) = \phi(s),$$
and therefore the original IVP has the implicitly defined solution

\[ u(x, t) = \phi(x - g(u)t). \]

In general, this equation can not be solved to obtain an explicit formula for \( u(x, t) \). However, it is still possible to understand the behavior of the solution by obtaining the equations of the characteristics and using the fact that the initial data is propagating along the curves.

Specifically, consider the characteristic that begins at \((x_0, 0)\). This curve has the equation

\[ t = \frac{x - x_0}{g(\phi(x_0))} \]

when \( g(\phi(x_0)) \neq 0 \); otherwise, it is simply the vertical line \( x = x_0 \). Along this curve, the solution is equal to \( \phi(x_0) \). However, for a nonlinear PDE, unlike the linear case, the characteristics may intersect, which causes a discontinuity in the solution. This is referred to as a shock. When this occurs, the discontinuity continues to propagate with speed

\[ \frac{dx}{dt} = \frac{f(u_R) - f(u_L)}{u_R - u_L}, \]

where \( u_L \) and \( u_R \) are the values of the solution on the left and right sides of the discontinuity, respectively.

**Transversality**

It is worth noting that the inversion of coordinates from \((\tau, s)\) to \((x, t)\) is not always possible. The dependence of the characteristics on \( \tau \) comes from the PDE, while the dependence on \( s \) comes from the initial conditions. Because the PDE and initial conditions are independent, it follows that for any PDE, there exists initial conditions for which the transformation from \((\tau, s)\) to \((x, t)\) is not invertible. This occurs if

\[ J = \frac{\partial(x, t)}{\partial(\tau, s)} = x_\tau t_s - t_\tau t_s = a(t_0) - b(x_0) = 0, \]

where \( J \) is the Jacobian of the transformation between the two sets of variables and the coefficients \( a \) and \( b \) are evaluated on the initial curve. When \( J \neq 0 \), we say that the IVP satisfies the transversality condition. Geometrically, this means that the initial curve cannot be tangent to a characteristic.