Examples of the Method of Characteristics

In this section, we present several examples of the method of characteristics for solving an IVP (initial value problem), without boundary conditions, which is also known as a Cauchy problem.

Example 1 We first solve the IVP
\[ u_x = 1, \quad u(0, y) = g(y) \]

The characteristic IVPs are
\[ x_\tau = 1, \quad x(0, s) = 0 \]
\[ y_\tau = 0, \quad y(0, s) = s \]
\[ u_\tau = 1, \quad u(0, s) = g(s) \]

The solutions of these IVPs are
\[ x(\tau, s) = \tau, \quad y(\tau, s) = s, \quad u(\tau, s) = \tau + g(s) \]

Inversion yields parametric forms of the characteristic curves
\[ s = y, \quad \tau = x, \quad u(x, t) = x + g(y) \]

Eliminating parameters, we see that the characteristics, which are simply the projections of the characteristic curves onto the \((x, y)\)-plane, are just horizontal lines \(y = y_0\), where \(y_0\) is an arbitrary constant.

Now, we solve the same PDE with an alternative initial condition
\[ u(x, 0) = h(x). \]

This time, the characteristic IVPs are
\[ x_\tau = 1, \quad x(0, s) = s \]
\[ y_\tau = 0, \quad y(0, s) = 0 \]
\[ u_\tau = 1, \quad u(0, s) = g(s) \]

The solutions are
\[ x(\tau, s) = \tau + s, \quad y(\tau, s) = 0, \quad u(\tau, s) = \tau + g(s) \]

Unfortunately, in this case, the inversion of the transformation from \((\tau, s)\) to \((x, y)\) cannot be inverted, because this IVP does not satisfy the transversality condition. We have
\[ J = a(y_0)s - b(x_0)s = 1(0) - 0(1) = 0, \]
so the Jacobian of the transformation is singular. Geometrically, we can see that inversion is not possible because the initial curve is the \(x\)-axis \(y = 0\), which is also a characteristic. Recall that the transversality condition implies that a characteristic cannot be tangent to the initial curve, let alone coincide with it. \(\blacksquare\)
For a linear PDE, as mentioned previously, the characteristics can be solved for independently of the solution $u$. Furthermore, the characteristic equations $x_\tau = a(x, y)$, $y_\tau = b(x, y)$ are autonomous, meaning that there is no explicit dependence on $\tau$, so the characteristics satisfy the ODE

$$\frac{dy}{dx} = \frac{dy/\tau}{dx/\tau} = \frac{b(x, y)}{a(x, y)}.$$  

For example, in the PDE $u_x + \sqrt{x} u_y = 0$,

the characteristics satisfy $dy/dx = \sqrt{x}$, which has the solution $y = \frac{2}{3} x^{3/2} + C$, where $C$ is an arbitrary constant.

**Example 2**

$$u_x + u_y + u = 1, \quad u(x, x + x^2) = \sin x, \quad x > 0.$$  

Characteristic IVPs:

$$x_\tau = 1, \quad x(0, s) = s,$$

$$y_\tau = 1, \quad y(0, s) = s + s^2,$$

$$u_\tau + u = 1, \quad u(0, s) = \sin s$$  

Solutions (characteristic curves):

$$x = \tau + s, \quad y = \tau + s + s^2, \quad u = 1 - e^{-\tau} + e^{-\tau} \sin s$$  

Inversion:

$$s = (y - x)^{1/2}, \quad \tau = x - (y - x)^{1/2}$$  

Solution:

$$u(x, y) = 1 - e^{-x+(y-x)^{1/2}[1 - \sin(y-x)^{1/2}]}$$  

□

**Example 3**

$$-yu_x + xu_y = u, \quad u(x, 0) = \psi(x)$$  

Characteristic IVPs:

$$x_\tau = -y, \quad x(0, s) = s,$$

$$y_\tau = x, \quad y(0, s) = 0,$$

$$u_\tau = u, \quad u(0, s) = \psi(s)$$  

Solutions (characteristic curves):

$$x = s \cos \tau, \quad y = s \sin \tau, \quad u = e^\tau \psi(s)$$  

Inversion:

$$s = \sqrt{x^2 + y^2}, \quad \tau = \tan^{-1} \frac{y}{x}$$  

Solution:

$$u(x, y) = \psi(\sqrt{x^2 + y^2}) \exp \left[ \tan^{-1} \frac{y}{x} \right]$$  

Although the characteristics intersect the initial curve (the $x$-axis) twice, the solution is still unique because we have chosen the value of $\tau$ that arises from choosing $x > 0$. □
Example 4

\[ u_x + 3y^{2/3}u_y = 2, \quad u(x, 1) = 1 + x \]

Characteristic IVPs:

\[ x_\tau = 1, \quad x(0, s) = s \]
\[ y_\tau = 3y^{2/3}, \quad y(0, s) = 1 \]
\[ u_\tau = 2, \quad u(0, s) = 1 + s \]

Characteristic curves:

\[ x = \tau + s, \quad y = (\tau + 1)^3, \quad u = 1 + s + 2\tau \]

Inversion:

\[ s = x + 1 - y^{1/3}, \quad \tau = y^{1/3} - 1 \]

Solution:

\[ u(x, y) = x + y^{1/3} \]

It is worth noting that \( y = 0 \) is also a characteristic, as it satisfies the characteristic IVP for \( y \). Because the characteristics we have computed intersect this “extra” characteristic, it follows that the solution has irregular behavior near \( y = 0 \); in fact, the solution is not differentiable there. \( \Box \)

Example 5 We now solve the quasilinear IVP

\[ (y + u)u_x + yu_y = x - y, \quad u(x, 1) = 1 + x \]

Characteristic IVPs:

\[ x_\tau = y + u, \quad x(0, s) = s \]
\[ y_\tau = y, \quad y(0, s) = 1 \]
\[ u_\tau = x - y, \quad u(0, s) = 1 + s \]

To obtain the characteristic curves, we first solve the IVP for \( y \):

\[ y = e^{\tau} \]

The IVPs for \( x \) and \( u \) are coupled, but if we let \( w = x + u \), we obtain

\[ w_\tau = w, \quad w(0, s) = 1 + 2s \]

which can easily be solved to obtain

\[ w = x + u = (1 + 2s)e^\tau. \]

If we also let \( v = x - u \), then we obtain

\[ v_\tau = 2y - v = 2e^\tau - v, \quad v(0, s) = -1, \]

which has the solution

\[ v = e^{-\tau}[e^{2\tau} - 2] = e^\tau - 2e^{-\tau}. \]

Solving the linear system

\[ x + u = w, \quad x - u = v \]
yields
\[ x = \frac{1}{2}(w + v) = (1 + s)e^\tau - e^{-\tau}, \quad u = \frac{1}{2}(w - v) = se^\tau + e^{-\tau}. \]

Checking the transversality condition
\[ J = a(y_0)_s - b(x_0)_s = (1 + 1 + s)(0) - 1(1) = -1 \neq 0 \]
shows that the change of coordinates from \((\tau, s)\) to \((x, y)\) can be inverted, to obtain
\[ s = x/y + 1/y^2 - 1, \quad y = \ln \tau, \]
though the solution \(u(x, t)\) can be obtained more easily by noting
\[ x = (1 + s)y - \frac{1}{y}, \quad u = sy + \frac{1}{y} \]
to obtain
\[ sy = x - y + \frac{1}{y} \]
from which we conclude
\[ u = x - y + \frac{2}{y}. \]

Note that the solution is not defined on the \(x\)-axis.