Conservation Laws and Shock Waves

Consider the conservation law
\[ u_t + \frac{\partial}{\partial x} F(u) = 0, \]
which can also be written as
\[ u_t + g(u)u_x = 0, \]
where \( g(u) = F'(u) \). We include the initial condition \( u(x,0) = h(x) \). Applying the method of characteristics, we obtain the IVPs
\[
\begin{align*}
x_{\tau} &= g(u(\tau,s)), \quad x(0,s) = s, \\
t_{\tau} &= 1, \quad t(0,s) = 0, \\
u_{\tau} &= 0, \quad u(0,s) = h(s)
\end{align*}
\]
which have the solutions
\[ x = s + g(h(s))\tau, \quad t = \tau, \quad u = h(s). \]

We conclude that the solution is represented implicitly by the equation
\[ u(x,t) = h(x - g(u)t). \]

The characteristics are defined by the equations
\[ x = x_0 + g(h(x_0))t. \]

That is, the characteristics are straight lines in the \((x,t)\) plane, with slope \(1/g(h(x_0))\). However, because the slopes depend on the initial points, it is possible that they might intersect. When this occurs, the solution develops a discontinuity.

To determine when this might occur, we compute \( u_x \) from the implicit form of the solution and obtain
\[ u_x = h'(x - g(u)t)(1 - g'(u)u_x t) \]
which yields
\[ u_x = \frac{h'(s)}{1 + h'(s)g'(u)t}. \]

It follows that the slope becomes infinite, thus causing a jump discontinuity in the solution, at the critical time
\[ t_c = -\frac{1}{\frac{d}{ds}[g(h(s))]} \]
Because we are only considering \( t > 0 \), we must have that the speed at which \( h(s) \) propagates along the characteristics, which is given by \( g(h(s)) \), is a decreasing function of \( s \).
How does the solution behave once the discontinuity forms? To answer that, we examine a weak solution obtained by integrating the PDE with respect to $x$ over an interval $[a, b]$, which yields

$$
\int_a^b u_t(\xi, t) \, d\xi + \int_a^b \frac{\partial}{\partial \xi} F(u(\xi, t)) \, d\xi = 0,
$$

which simplifies to

$$
\frac{\partial}{\partial t} \int_a^b u(\xi, t) \, d\xi + F(u(b, t)) - F(u(a, t)) = 0.
$$

Suppose the discontinuity develops along a curve $x = \gamma(t)$. Then we have

$$
\frac{\partial}{\partial t} \left[ \int_a^\gamma u(\xi, t) \, d\xi + \int_\gamma^b u(\xi, t) \, d\xi \right] + F(u(b, t)) - F(u(a, t)) = 0.
$$

Let $u^-$ and $u^+$ be the values of $u$ on the left and right sides, respectively, of the discontinuity at a fixed time. Differentiating with respect to $t$ yields

$$
u^- \gamma'(t) - u^+ \gamma'(t) + \int_a^\gamma u_t(\xi, t) \, d\xi + \int_\gamma^b u_t(\xi, t) \, d\xi + F(u(b, t)) - F(u(a, t)) = 0.
$$

Using the PDE inside the integrals, we obtain

$$
u^- \gamma'(t) - u^+ \gamma'(t) - \int_a^\gamma F(u(\xi, t)) \, d\xi - \int_\gamma^b F(u(\xi, t)) \, d\xi + F(u(b, t)) - F(u(a, t)) = 0.
$$

Evaluating the integrals and simplifying yields

$$
\gamma'(t) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}.
$$

That is, the jump discontinuity propagates at a speed that is equal to the average rate of change of $F$ from $u^-$ to $u^+$. The above equation is called the Rankine-Hugoniot condition.

**Example 1** Consider the IVP

$$
u_t + uu_x = 0,
$$

$$
u(x, 0) = h(x) = \begin{cases} 
1 & x \leq 0 \\
1 - x/\alpha & 0 < x < \alpha \\
0 & x \geq \alpha
\end{cases}.
$$

From this definition of $h(x)$, we obtain the critical time $t_c = \alpha$. For $t < \alpha$, using the implicit equation defining the solution, $u = h(x - vt)$, we obtain

$$
u(x, t) = \begin{cases} 
1 & x \leq t \\
\frac{t}{t-\alpha} & t < x < \alpha \\
0 & x \geq \alpha
\end{cases}.
$$

For $t \geq t_c$, using the Rankine-Hugoniot condition with $F(u) = \frac{1}{2}u^2$, we obtain

$$
\gamma'(t) = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}(u^+ - u^-) = \frac{1}{2}(0 + 1) = \frac{1}{2}.
$$

This means that the discontinuity propagates to the right with speed $\frac{1}{2}$. It follows that for $t > \alpha$, the PDE has the weak solution

$$u(x,t) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(t - \alpha) \\ 0 & x > \alpha + \frac{1}{2}(t - \alpha) \end{cases}.$$  

This solution is a step function moving to the right with speed $\frac{1}{2}$. Such a solution is called a **shock wave**. Because it arises from the intersection of characteristics, which give the solution the appearance of contracting, it is also known as a **compression wave**. □

**Example 2** Now, consider the same PDE, but with increasing initial data:

$$u(x,0) = h(x) = \begin{cases} 0 & x \leq 0 \\ x/\alpha & 0 < x < \alpha \\ 1 & x \geq \alpha \end{cases}.$$  

There is no positive critical time, so the solution, that is valid for all $t > 0$, is

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t+\alpha} & 0 < x < t + \alpha \\ 1 & x \geq t + \alpha \end{cases}.$$  

This solution is called an **expansion wave**, as the graph appears to expand over time. □

How can we tell whether the solution will be a compression wave or an expansion wave? A shock can only develop if there is a positive critical time $t_c$, which is only the case if the propagation speed, $g(h(s))$, is a decreasing function of $s$. This allows characteristics emanating from smaller values of $s$ to “catch up” to those emanating from larger values of $s$, when $F$ is increasing which causes propagation to the right. When $F$ is decreasing, meaning that the initial data is propagated to the left, characteristics emanating from larger values of $s$ catch up to those emanating from smaller values of $s$.

In either case, we must have, for each fixed $t$,

$$F'(u^-) > \gamma'(t) > F'(u^+).$$  

This is called the **entropy condition**, and it ensures a unique solution in the case of an expansion wave (otherwise, there is a weak solution that has a shock). Geometrically, the entropy condition states that characteristics can only enter a shock curve $x = \gamma(t)$; they cannot emanate from it. Because the initial data is propagated along characteristics, which emanate from the initial curve, another interpretation of the entropy condition is that as time increases, information can only be lost; it cannot be created.