(1) Let $V$ be the vector space of all bounded or unbounded sequences of complex numbers.
(a) Define $d : V \times V \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}.$$ 

Show that $d$ is a metric on $V$.

**Solution:** Let $x = (\xi_j), y = (\eta_j) \in V$. For any $j = 1, 2, \cdots$,

$$0 \leq \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \frac{1}{2^j}$$

and $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges (geometric series). So by comparison test, $d(x, y) < \infty$ for all $x, y \in V$. Clearly then $d(x, y) \in \mathbb{R}^+ \cup \{0\}$. Also clearly $d(x, y) = d(y, x)$ for all $x, y \in V$ i.e. (M2) is satisfied. If $x = y$ then $\xi_j = \eta_j$ for all $j = 1, 2, \cdots$ and so $d(x, y) = 0$. Suppose that $d(x, y) = 0$. Then for any $j = 1, 2, \cdots$,

$$0 \leq \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq d(x, y) = 0$$

$$\Rightarrow \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} = 0, \forall j = 1, 2, \cdots$$

$$\Rightarrow \xi_j = \eta_j, \forall j = 1, 2, \cdots$$

$$\Rightarrow x = y.$$ 

Thus, (M1) is satisfied. Now we show that the Triangle Inequality (M3). Let $x = (\xi_j), y = (\eta_j)$ and $z = (\zeta_j)$. \[1\]
Then for each $j = 1, 2, \ldots$, 

$$
\frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j| + |\xi_j - \eta_j|} \leq \frac{|\xi_j - \zeta_j|}{1 + |\xi_j - \zeta_j|},
\frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|} \leq \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.
$$

Recall that if $0 \leq a \leq b$ then 

$$
\frac{a}{1+a} \leq \frac{b}{1+b}.
$$

By triangle inequality with respect to $| \cdot |$, we have 

$$
|\xi_j - \eta_j| \leq |\xi_j - \zeta_j| + |\zeta_j - \eta_j|
$$

for each $j = 1, 2, \ldots$. It then follows from the inequality (1) that 

$$
\frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|} \leq \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|} + \frac{|\zeta_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.
$$

So, for each $n = 1, 2, \ldots$ we have 

$$
\sum_{j=1}^{n} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|} \leq \sum_{j=1}^{n} \frac{1}{2^j} \frac{|\xi_j - \zeta_j| + |\zeta_j - \eta_j|}{1 + |\xi_j - \zeta_j| + |\zeta_j - \eta_j|} = \sum_{j=1}^{n} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\zeta_j - \eta_j|}.
$$

Finally by taking the limit $n \to \infty$, we obtain the Triangle Inequality (M3).

(b) Show that $d$ is not translation invariant.

**Solution:** Let $a = (a_j) \in V$. Then 

$$
d(x+a, y+a) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j + a_j - (\eta_j + a_j)|}{1 + |\xi_j + a_j - (\eta_j + a_j)|} = d(x, y).
$$
Let $\alpha$ be a scalar. Then

$$d(\alpha x, \alpha y) = \sum_{j=1}^{\infty} \frac{|\alpha \xi_j - \alpha \eta_j|}{1 + |\alpha \xi_j - \alpha \eta_j|}$$

$$= \sum_{j=1}^{\infty} \frac{\alpha |\xi_j - \eta_j|}{1 + \alpha |\xi_j - \eta_j|}$$

and this does not necessarily coincide with $d(x, y)$. To see this, for example let $x = (1, 1, 1, \ldots)$, $y = (0, 0, 0, \ldots)$ and $\alpha = 2$. Then

$$d(\alpha x, \alpha y) = \sum_{j=1}^{\infty} \frac{1}{2^j \frac{2}{3}}$$

$$= \frac{1}{3} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}}$$

$$= \frac{1}{3} \left(1 - \frac{1}{2}\right)$$

$$= \frac{2}{3},$$

while

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} = 1.$$

Hence, $d$ is not translation invariant. Therefore, the metric $d$ cannot be induced by a norm on a normed space.

(2) Show that in a normed space $X$, vector addition and multiplication by scalars are continuous operations with respect to the norm. That is, the mappings defined by $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$ are continuous.

**Solution:** Let $a, b \in X$. Let $\varepsilon > 0$ be given. Choose $\delta_1 = \delta_2 = \frac{\varepsilon}{2}$. Then whenever $||x - a|| < \delta_1$ and $||y - b|| < \delta_2$,

$$||x + y - (a + b)|| \leq ||x - a|| + ||y - b|| < \varepsilon.$$
Hence, the addition $+$ is continuous. Let $\alpha$ be a scalar and $a \in X$. Let $\epsilon > 0$ be given. If $\alpha = 0$, then $\|\alpha x - \alpha a\| = 0 < \epsilon$.

Now suppose $\alpha \neq 0$. Choose $\delta = \frac{\epsilon}{|\alpha|} > 0$. Then

$$\|\alpha x - \alpha a\| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon.$$ 

Hence, the scalar multiplication is continuous.

(3) Show that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$. Show that $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$.

**Solution:** Assume that $x_n \to x$ and $y_n \to y$. Let $\epsilon > 0$ be given. Then there exists positive integers $N_1 > 0$ and $N_2 > 0$ such that

$$\|x_n - x\| < \frac{\epsilon}{2}$$

for all $n \leq N_1$ and

$$\|y_n - y\| < \frac{\epsilon}{2}$$

for all $n \geq N_2$. Choose $N = \max\{N_1, N_2\}$. Then for all $n \geq N$

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

Assume that $\alpha_n \to \alpha$ and $x_n \to x$. Let $\epsilon > 0$ be given. Since $(x_n)$ is convergent, there exists $M > 0$ such that $\|x_n\| < M$. If $\alpha = 0$, then there exists a positive integer $N$ such that $|\alpha_n| < \frac{\epsilon}{M}$ for all $n \geq N$. So, for all $n \geq N$

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n\| = \|x_n\||\alpha_n\| < M \frac{\epsilon}{M} = \epsilon.$$ 

Now suppose that $\alpha \neq 0$. Then there exists a positive integer $N_1$ such that

$$|\alpha_n - \alpha| < \frac{\epsilon}{2M}$$
for all \( n \geq N_1 \) and there exists a positive integer \( N_2 \) such that

\[
||x_n - x|| < \frac{\epsilon}{2|\alpha|}
\]

for all \( n \geq N_2 \). Choose \( N = \max\{N_1, N_2\} \). Then for all \( n \geq N \)

\[
||\alpha_n x_n - \alpha x|| = ||\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x||
\]

\[
\leq |\alpha_n - \alpha||x_n|| + |\alpha||x_n - x||
\]

\[
< \frac{\epsilon}{2M} + |\alpha|\frac{\epsilon}{2|\alpha|}
\]

\[
= \epsilon.
\]

(4) Show that the closure \( \bar{Y} \) of a subspace \( Y \) of a normed space \( X \) is again a vector subspace.

**Solution:** Let \( y_1, y_2 \in \bar{Y} \). Then there exist sequences \((\xi_n), (\eta_n) \subset Y\) such that \( \xi_n \to y_1 \) and \( \eta_n \to y_2 \). Then by \#3, \( \xi_n + \eta_n \to y_1 + y_2 \). Since \( (\xi_n + \eta_n) \subset Y \), \( y_1 + y_2 \in \bar{Y} \). Also by \#3, \( \alpha \xi_n \to \alpha y_1 \). Since \( (\alpha \xi_n) \subset Y \), \( \alpha y_1 \in \bar{Y} \).

(5) In a normed space, convergence of a series implies absolute convergence of that series. But the converse need not be true. In fact, we see, throughout the questions (a)-(c), that in a normed space \( X \) absolute convergence of a series implies convergence of that series if and only if \( X \) is complete.

(a) Show that in a normed space absolute convergence of a series does not necessarily imply convergence of that series.

**Solution:** Let \( Y \) be the set of all sequences with only finitely many nonzero complex terms. Then clearly \( Y \) is a subset of \( \ell^\infty \). Let \( (y_n) \) be a sequence such that

\[
y_1 = (1, 0, 0, \cdots),
\]

\[
y_2 = \left(0, \frac{1}{2^2}, 0, \cdots\right),
\]

\[
y_3 = \left(0, 0, \frac{1}{3^2}, 0, \cdots\right),
\]

\cdots
Then \((y_n)_n \subset Y\).

\[
\|y_1\| + \|y_2\| + \|y_3\| + \cdots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

So, \(\sum_{n=1}^{\infty} y_n\) converges absolutely. However,

\[
y_1 + y_2 + y_3 + \cdots = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots\right) \not\subseteq Y,
\]

i.e. \(\sum_{n=1}^{\infty} y_n\) does not converge.

(b) If in a normed space \(X\), absolute convergence of any series always implies convergence of that series, show that \(X\) is complete.

**Solution:** Let \((x_n)_n \subset X\) be a Cauchy sequence. Let

\[
a_1 = x_2 - x_1, \\
a_2 = x_3 - x_2, \\
\vdots \\
a_n = x_{n+1} - x_n, \\
\vdots
\]

and let \(b_n = \|a_n\|, n = 1, 2, \cdots\). Also let \(s'_n\) denote the \(n\)-th partial sum of the \(b_n\)'s. For \(m < n\)

\[
|s'_n - s'_m| = |b_n + b_{n-1} + \cdots + b_{m+1}|
= \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_{m+2} - x_{m+1}\|.
\]

Since \(m < n, n = m + l\) for some positive integer \(l\). Let \(\epsilon > 0\) be given. Then there exists a positive integer \(N\) such that

\[
\|x_n - x_m\| < \frac{\epsilon}{l}
\]

for all \(m, n \geq N\). So, for all \(m,n \geq N\),

\[
|s'_n - s'_m| < \frac{\epsilon}{l} + \cdots + \frac{\epsilon}{l} = \epsilon,
\]
i.e. \((s'_n)\) is a Cauchy sequence in \(\mathbb{R}\). Since \(\mathbb{R}\) is complete, \((s'_n)\) is a convergent sequence:

\[
\sum_{k=1}^{\infty} ||a_k|| = \lim_{n \to \infty} s'_n < \infty.
\]

That is, \(\sum_{k=1}^{\infty} a_k\) is absolutely convergent. By assumption, it is convergent. Let us denote the \(n\)-th partial sum of the \(a_n\)’s by \(s_n\). Then

\[
\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n
\]

\[
= \lim_{n \to \infty} [(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{n+1} - x_n)]
\]

\[
= \lim_{n \to \infty} (x_{n+1} - x_1) < \infty.
\]

Hence, \((x_n)\) is a convergent sequence in \(X\). Now we have shown that any Cauchy sequence in \(X\) is a convergent sequence and therefore \(X\) is complete.

(c) Show that in a Banach space, an absolutely convergent series is convergent.

**Solution:** Let \(\sum_{n=1}^{\infty} x_n\) be an absolutely convergent series. For \(m < n\),

\[
||s_n - s_m|| = ||x_{m+1} + \cdots + x_n||
\]

\[
\leq ||x_{m+1}|| + \cdots + ||x_n||
\]

\[
\leq ||x_{m+1}|| + \cdots
\]

\[
= \sum_{n=1}^{\infty} ||x_n|| - (||x_1|| + \cdots + ||x_m||)
\]

\[
\to 0
\]

as \(m \to \infty\). So, \((s_n)\) is a Cauchy sequence in the Banach space and hence it is convergent.

(6) Show that \((e_n)\), where \(e_n = (\delta_n)\), is a Schauder basis for \(\ell^p\), where \(1 \leq p < \infty\).
Solution: Let $x = (\xi_j) \in \ell^p$. Then

$$x = (\xi_1, \xi_2, \xi_3, \cdots)$$

$$= \xi_1(1, 0, 0, \cdots) + \xi_2(0, 1, 0, \cdots) + \xi_3(0, 0, 1, 0, \cdots) + \cdots$$

$$= \xi e_1 + \xi e_2 + \xi e_3 + \cdots$$

and $e_1, e_2, e_3, \cdots \in \ell^p$. Hence, $(e_n)$ is a Schauder basis for $\ell^p$. 