Surfaces of Revolution in Hyperbolic 3-Space

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Surfaces of Constant Mean Curvature in Hyperbolic 3-Space
Parametric Surfaces in Hyperbolic 3-Space
Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$
The Illustration of the Limit of Surfaces of Revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$
Minimal Surface of Revolution in $\mathbb{H}^3(-c^2)$

Outline

1. Surfaces of Constant Mean Curvature in Hyperbolic 3-Space
2. Parametric Surfaces in Hyperbolic 3-Space
3. Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$
4. The Illustration of the Limit of Surfaces of Revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$
5. Minimal Surface of Revolution in $\mathbb{H}^3(-c^2)$
Hyperbolic 3-Space $\mathbb{H}^3(-c^2)$

- Let $\mathbb{R}^{3+1}$ denote the Minkowski spacetime with Lorentzian metric
  \[ ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \]

- Hyperbolic 3-space $\mathbb{H}^3(-c^2)$ is the hyperquadric defined by
  \[ -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\frac{1}{c^2}. \]

- $\mathbb{H}^3(-c^2)$ has the constant sectional curvature $-c^2$. 
Hyperbolic 3-Space $\mathbb{H}^3(-c^2)$

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Pseudospherical Model

- On the chart

\[ U = \{(x^0, x^1, x^2, x^3) \in \mathbb{H}^3(-c^2) : x^0 + x^1 > 0\} \]

define

\[ t = -\frac{1}{c} \log c(x^0 + x^1), \]

\[ x = \frac{x^2}{c(x^0 + x^1)}, \]

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\[ ds^2 = (dt)^2 + e^{-2ct} \left\{ (dx)^2 + (dy)^2 \right\} \]
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- \[ ds^2 = (dt)^2 + e^{-2ct} \left\{ (dx)^2 + (dy)^2 \right\} \]
\[ \mathbb{R}^3 \text{ with coordinates } t, x, y \text{ and the metric} \]

\[ g_c = (dt)^2 + e^{-2ct} \left\{ (dx)^2 + (dy)^2 \right\} \]

is called the \textit{pseudospherical model} of hyperbolic 3-space.

- The pseudospherical model is a local chart of \( \mathbb{H}^3(-c^2) \), so it is not regarded as one of the standard models of hyperbolic 3-space.

- As \( c \to 0 \), \( (\mathbb{R}^3, g_c) \) flattens out to Euclidean 3-space \( \mathbb{E}^3 \).
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$\mathbb{R}^3$ with coordinates $t, x, y$ and the metric

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Pseudospherical Model

Continued

- $(\mathbb{R}^3, g_c)$ is isometric to a solvable Lie group $G_c$ with a left-invariant metric

$$G_c = \left\{ \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & e^{ct} & 0 & x \\ 0 & 0 & e^{ct} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t, x, y) \in \mathbb{R}^3 \right\}.$$


- From here on, we will denote $(\mathbb{R}^3, g_c)$ simply by $\mathbb{H}^3(-c^2)$. 

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Surfaces of Revolution
Lawson Correspondence

- There is an interesting correspondence, called *Lawson correspondence*, between constant mean curvature surfaces in different Riemannian space forms. H. Blain Lawson, Jr., *Complete minimal surfaces in S*³, *Ann. of Math.* 92, 335-374 (1970)

- Those corresponding constant mean curvature surfaces satisfy the same Gauß-Codazzi equations, so they share many geometric properties in common.

- There is a one-to-one correspondence between surfaces of constant mean curvature \( H_h \) in \( \mathbb{H}^3(−c^2) \) and surfaces of constant mean curvature \( H_e = ±\sqrt{H_h^2 − c^2} \) in \( \mathbb{E}^3 \).
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Lawson Correspondence
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- In particular, surfaces of constant mean curvature $H = \pm c$ in $\mathbb{H}^3(-c^2)$ are cousins of minimal surfaces in $\mathbb{E}^3$.

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Surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ can be constructed with a holomorphic and a meromorphic data using Bryant’s representation formula, analogously to Weierstraß representation formula for minimal surfaces in $\mathbb{E}^3$.


However, it is not suitable for constructing surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$. 
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Conformal Parametric Surfaces in $\mathbb{H}^3(-c^2)$

Definition

A parametric surface $\varphi : M \rightarrow \mathbb{H}^3(-c^2)$ is said to be *conformal* if

$$\langle \varphi_u, \varphi_v \rangle = 0, |\varphi_u| = |\varphi_v| = e^{\omega/2},$$

where $(u, v)$ is a local coordinate system in $M$ and $\omega : M \rightarrow \mathbb{R}$ is a real-valued function in $M$.

The induced metric on the conformal parametric surface is given by

$$ds^2_{\varphi} = e^{\omega} \left\{ (du)^2 + (dv)^2 \right\}.$$
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Cross Product in $T_p \mathbb{H}^3(-c^2)$

- $\mathbb{H}^3(-c^2)$ is not a vector space but each tangent space $T_p \mathbb{H}^3(-c^2)$ is, and we can consider cross product on each $T_p \mathbb{H}^3(-c^2)$.

- For $v = v_1 \left( \frac{\partial}{\partial t} \right)_p + v_2 \left( \frac{\partial}{\partial x} \right)_p + v_3 \left( \frac{\partial}{\partial y} \right)_p$,
  $w = w_1 \left( \frac{\partial}{\partial t} \right)_p + w_2 \left( \frac{\partial}{\partial x} \right)_p + w_3 \left( \frac{\partial}{\partial y} \right)_p \in T_p \mathbb{H}^3(-c^2)$, define
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Cross Product in $T_p\mathbb{H}^3(-c^2)$

Continued

**Definition**

The cross product $\mathbf{v} \times \mathbf{w}$ is defined by

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \left( \frac{\partial}{\partial t} \right)_p + e^{2ct} (v_3 w_1 - v_1 w_3) \left( \frac{\partial}{\partial x} \right)_p + e^{2ct} (v_1 w_2 - v_2 w_1) \left( \frac{\partial}{\partial y} \right)_p,$$

where $p = (t, x, y) \in \mathbb{H}^3(-c^2)$. 
The Mean Curvature of a Conformal Parametric Surface in $\mathbb{H}^3(-c^2)$

If a parametric surface $\varphi : M \rightarrow \mathbb{H}^3(-c^2)$ is conformal, the mean curvature $H$ is computed by the formula

$$H = \frac{G\ell + En - 2Fm}{2(EG - F^2)},$$

where

$$E = \langle \varphi_u, \varphi_u \rangle, \quad F = \langle \varphi_u, \varphi_v \rangle, \quad G = \langle \varphi_v, \varphi_v \rangle$$

$$\ell = \langle \varphi_{uu}, N \rangle, \quad m = \langle \varphi_{uv}, N \rangle, \quad n = \langle \varphi_{vv}, N \rangle$$

and $N = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}$ is a unit normal vector field on $\varphi$. 
Rotations in $\mathbb{H}^3(-c^2)$

- Rotations about the $t$-axis are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$.

- The rotation of a profile curve $\alpha(u) = (u, h(u), 0)$ in the $tx$-plane about the $t$-axis through an angle $\nu$:

$$\varphi(u, \nu) = (u, h(u) \cos \nu, h(u) \sin \nu).$$
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Differential Equation of $h(u)$ for Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$

- The mean curvature $H$ of a conformal surface of revolution in $\mathbb{H}^3(-c^2)$ is computed to be

$$H = \frac{-h''(u) + h(u)}{2e^{-2cu}(h(u))^3}.$$ 

- By setting $H = c$, we obtain the second order non-linear differential equation of $h(u)$

$$h''(u) - h(u) + 2ce^{-2cu}(h(u))^3 = 0.$$
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Minimal Surface of Revolution in $\mathbb{H}^3(-c^2)$
Questions
Limit Behavior of Surfaces of Revolution with CMC $H = c$ as $c \to 0$

- If $c \to 0$, then the differential equation of $h(u)$ becomes
  \[ h''(u) - h(u) = 0, \]
  which is a harmonic oscillator. Its solution is
  \[ h(u) = c_1 \cosh u + c_2 \sinh u. \]
- For $c_1 = 1$, $c_2 = 0$, we obtain the catenoid
  \[ \varphi(u, \nu) = (u, \cosh u \cos \nu, \cosh u \sin \nu), \]
  the minimal surface of revolution in $\mathbb{H}^3$. 
Limit Behavior of Surfaces of Revolution with CMC $H = c$ as $c \to 0$

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Minimal Surface of Revolution in $\mathbb{H}^3(-c^2)$.

Questions:
Surface of Revolution with CMC $H = 1$ in $\mathbb{H}^3(-1)$

Figure: CMC $H = 1$; Profile Curve

Surfaces of Revolution in Hyperbolic 3-Space
Parametric Surfaces in Hyperbolic 3-Space
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Surface of Revolution with CMC $H = 1$ in $\mathbb{H}^3(-1)$

Continued
Surface of Revolution with CMC $H = \frac{1}{4}$ in $\mathbb{H}^3(-\frac{1}{16})$
Surface of Revolution with CMC $H = \frac{1}{8}$ in $\mathbb{H}^3\left(-\frac{1}{64}\right)$
Surface of Revolution with CMC $H = \frac{1}{256}$ in $\mathbb{H}^3\left(-\frac{1}{65536}\right)$
Animations

- Animation of Profile Curves $h(u)$
  http://www.math.usm.edu/lee/profileanim.gif

- Animation of Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$
  http://www.math.usm.edu/lee/cmcanim.gif
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Harmonic Maps and Minimal Surfaces in $\mathbb{E}^3$

**Definition**

A smooth map $\varphi : M \rightarrow \mathbb{E}^3$ is harmonic if it is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2} \int_M ||d\varphi||^2$$

under every compactly supported variation of $\varphi$.

- $\varphi : M \rightarrow \mathbb{E}^3$ is harmonic if and only if $\triangle \varphi = 0$ where $
\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is Laplacian.

- A conformal surface $\varphi : M \rightarrow \mathbb{E}^3$ is minimal if and only if it is harmonic i.e. $\triangle \varphi = 0$. 

Surfaces of Revolution
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Surfaces of Revolution
For any conformal surface $\phi : M \rightarrow \mathbb{E}^3$, the mean curvature $H$ is computed to be

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Harmonic Maps and Minimal Surfaces in $\mathbb{E}^3$

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Minimal Surfaces in $\mathbb{H}^3(-c^2)$

- In $\mathbb{H}^3(-c^2)$, there is no relationship between minimal surfaces and mean curvature since harmonic map equation is no longer Laplace’s equation.

- Minimal surfaces in $\mathbb{H}^3(-c^2)$ can be in general constructed by Kokubu’s representation formula.

- However it is not suitable for constructing minimal surface of revolution in $\mathbb{H}^3(-c^2)$. 
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Construction of Minimal Surface in $\mathbb{H}^3(-c^2)$

- The area functional of $\varphi : M \rightarrow \mathbb{H}^3(-c^2)$ is

$$J = \int_{t_1}^{t_2} f(x, x_t, t) dt = \int_{t_1}^{t_2} 2\pi x \sqrt{1 + \left( \frac{dx}{dt} \right)^2} dt.$$ 

- The Euler-Lagrange equation $\frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial x_t} = 0$ is

$$\frac{d^2 x(t)}{dt^2} - 2 \frac{dx(t)}{dt} - x(t) - e^{-2ct} \left( \frac{dx(t)}{dt} \right)^3 = 0.$$
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Minimal Surface of Revolution in $\mathbb{H}^3(-1)$

Figure: Minimal Surface of Revolution in $\mathbb{H}^3(-1)$: Profile Curve
Minimal Surface of Revolution in $\mathbb{H}^3(-1)$

Continued

Figure: Minimal Surface of Revolution in $\mathbb{H}^3(-1)$
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Questions

Any Questions?