These notes correspond to Section 2.5 in the text.

**The Two-Body Problem**

We revisit the *two-body problem*

\[
m_1 r_1'' = -\frac{G m_1 m_2}{r_{12}^3} (r_2 - r_1),
\]

\[
m_2 r_2'' = -\frac{G m_1 m_2}{r_{12}^3} (r_1 - r_2),
\]

that describes the motion of two bodies, of mass \(m_1\) and \(m_2\) located at positions \(r_1\) and \(r_2\) due to the gravitational force they exert on each other.

**Conservation of Linear Momentum**

Adding equations (1) and (2) yields

\[
m_1 r_1'' + m_2 r_2'' = 0,
\]

which, upon integration with respect to \(t\), yields

\[
m_1 r_1' + m_2 r_2' = P,
\]

where \(P\) is the *total linear momentum* of the system, given by

\[
P = m_1 r_1'(0) + m_2 r_2'(0).
\]

Integrating a second time with respect to \(t\) yields

\[
m_1 r_1 + m_2 r_2 = Pt + B,
\]

where \(B\) is a constant vector.

**Uniform Motion of the Center of Mass**

Let \(M = m_1 + m_2\) denote the *total mass* of the system. From (4), we obtain

\[
\frac{m_1}{M} r_1 + \frac{m_2}{M} r_2 = V t + C.
\]

The left side of this equation is the *center of mass* of the system. We see that the center of mass moves with constant velocity \(V = P/M\) and has initial position \(C = B/M\).
Jacobi Coordinates

The following change of variables provides a useful alternative perspective on the two-body problem.

**Definition 1 (Jacobi Coordinates)** Let \( r_1 \) and \( r_2 \) be the solutions of the two-body system (1), (2). The variables \( R \) and \( r \), defined by

\[
R = \frac{m_1}{M} r_1 + \frac{m_2}{M} r_2, \tag{5}
\]

\[
r = r_2 - r_1, \tag{6}
\]

are called the **Jacobi coordinates** of the system.

From (1), (2), (5) and (6), we obtain

\[
R'' = 0, \tag{7}
\]

\[
r'' = r_2'' - r_1'' = -\frac{Gm_1}{r_{12}^3} r - \frac{Gm_2}{r_{12}^3} r = -\frac{GM}{r^3} r, \tag{8}
\]

where we use the shorthand \( r = \|r_2 - r_1\| = r_{12} \). A significant advantage of the system (7), (8) in Jacobi coordinates, compared to (1), (2) in Cartesian coordinates, is that the equations are **uncoupled**; that is, (7) depends only on \( R \) while (8) depends only on \( r \).

We have seen that (7) has the solution \( R = V t + C \); what remains is to solve (8). Once we do, we can use the fact that the Jacobi and Cartesian coordinates are related by the system of linear equations

\[
\begin{bmatrix}
R \\
r
\end{bmatrix} = \begin{bmatrix}
\frac{m_1}{M} & \frac{m_2}{M} \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix},
\]

which can be solved to obtain

\[
\begin{bmatrix}
r_1 \\
r_2
\end{bmatrix} = \frac{1}{M + \frac{m_2}{M}} \begin{bmatrix}
1 & -\frac{m_2}{M} \\
1 & \frac{m_1}{M}
\end{bmatrix} \begin{bmatrix}
R \\
r
\end{bmatrix} = \begin{bmatrix}
R - \frac{m_2}{M} r \\
R + \frac{m_2}{M} r
\end{bmatrix}.
\]

**The Central Force Problem**

The general **central force problem** is

\[
mr'' = -\frac{f(r)}{r} r,
\]

where the function \( f(r) \) describes the dependence of the force on the distance \( r \) between the two bodies, the first of which is located at the origin.

For the two-body problem we are considering, if we define the **reduced mass**

\[
m = \frac{m_1 m_2}{m_1 + m_2},
\]

then equation (8) can be rewritten as a central force problem

\[
m r'' = -\frac{Gm_1 m_2}{r^3} r,
\]

where \( f(r) = Gm_1 m_2 / r^2 \).
Conservation of Angular Momentum

Consider the vector 
\[ \mathbf{r} \times m \mathbf{r}', \]
which is normal to the plane spanned by the position vector \( \mathbf{r} \) and the tangent vector \( \mathbf{r}' \). Differentiating yields
\[
\frac{d}{dt} (\mathbf{r} \times m \mathbf{r}') = \mathbf{r}' \times m \mathbf{r}' + \mathbf{r} \times m \mathbf{r}'' = \mathbf{r} \times m \left( -\frac{f(r)}{r} \right) \mathbf{r} = 0
\]
That is, we have
\[
\mathbf{r} \times m \mathbf{r}' = \mathbf{L},
\]
where \( \mathbf{L} \) is a constant vector. This is conservation of angular momentum, which means that the motion of the body remains in the same plane.

Central Force Equations in Polar Coordinates

We now convert the central force problem
\[
\mathbf{r}'' = -\frac{kr}{m} \mathbf{r},
\]
where for convenience we have defined \( k = 1/m \), into polar coordinates \( x = r \cos \theta, y = r \sin \theta \). We begin by writing this system in terms of components,
\[
x'' = -\frac{kr}{r} x, \quad (9)
y'' = -\frac{kr}{r} y, \quad (10)
\]
where \( r = \sqrt{x^2 + y^2} \). Differentiating \( x = r \cos \theta \) and \( y = r \sin \theta \) with respect to \( t \) yields
\[
x' = r' \cos \theta - r \sin \theta \theta',
y' = r' \sin \theta + r \cos \theta \theta'.
\]
Differentiating with respect to \( t \) a second time yields
\[
x'' = r'' \cos \theta - 2r' \sin \theta \theta' - r \cos \theta \theta'' - r \sin \theta \theta''',
y'' = r'' \sin \theta + 2r' \cos \theta \theta' - r \sin \theta \theta'' + r \cos \theta \theta'''.
\]
Substituting these second derivatives into (9), (10) yields
\[
(r'' - r(\theta')^2) \cos \theta - (2r' \theta' + r \theta'') \sin \theta = -kf(r) \cos \theta,
y'' - r(\theta')^2 \sin \theta + (2r' \theta' + r \theta'') \cos \theta = -kf(r) \sin \theta.
\]
Now, if we multiply the first equation above by \( \cos \theta \) and the second equation by \( \sin \theta \), and add them, we obtain
\[
r'' - r(\theta')^2 = -kf(r). \quad (11)
\]
Kepler’s Second Law

Similarly, if we multiply the first equation by \( \sin \theta \) and the second equation by \( \cos \theta \), and then subtract them, we obtain

\[
2r'\theta' + r\theta'' = 0,
\]

which, after multiplying by \( r \), can be simplified to

\[
\frac{d}{dt} (r^2 \theta') = 0.
\]

Integrating yields Kepler’s Second Law,

\[
r^2 \theta' = c,
\]

where \( c \) is a constant. To interpret this law, we integrate both sides of this equation with respect to \( t \), and obtain

\[
\frac{1}{2} \int_{t_1}^{t_2} r^2 \theta'(t) \, dt = \frac{1}{2} c(t_2 - t_1),
\]

where the left side is the area swept out by the polar curve \( r = r(t) \) from \( \theta_1 = \theta(t_1) \) to \( \theta_2 = \theta(t_2) \). That is, equal areas are swept out in equal times.

To solve (11), we define the new variable \( \rho \) by \( r = 1/\rho(\theta) \). Using \( f(r) = Gm_1m_2/r^2 \), Kepler’s Second Law \( r^2 \theta' = c \), and the derivatives

\[
r' = \left( \frac{1}{\rho} \right)' = -\frac{\rho'(\theta)\theta'}{\rho^2} = -c\rho', \quad r'' = -c\rho''\theta' = -c^2 \rho'' \rho^2,
\]

we obtain the equation

\[
\rho'' + \rho = a,
\]

where \( a = k/c^2 \). It follows from the method of undetermined coefficients that

\[
\rho(\theta) = a + A \cos \theta + B \sin \theta,
\]

and therefore

\[
r(t) = \frac{1}{a + A \cos \theta(t) + B \sin \theta(t)},
\]

where the constants \( a, B \) and \( C \) can be determined from the initial conditions. To solve for \( \theta(t) \), we note that Kepler’s Second Law, when rewritten as

\[
\theta'(t) = c\rho(\theta(t))^2,
\]

is a separable ODE, which can be solved by integration.

Exercises

Section 2.5: Exercises 2, 3