Normed Linear Spaces

Previously we have considered the problem of polynomial interpolation, in which a function \( f(x) \) is approximated by a polynomial \( p_n(x) \) that agrees with \( f(x) \) at \( n+1 \) distinct points, based on the assumption that \( p_n(x) \) will be, in some sense, a good approximation of \( f(x) \) at other points. As we have seen, however, this assumption is not always valid, and in fact, such an approximation can be quite poor, as demonstrated by Runge’s example.

Therefore, we consider an alternative approach to approximation of a function \( f(x) \) on an interval \([a, b]\) by a polynomial, in which the polynomial is not required to agree with \( f \) at any specific points, but rather approximate \( f \) well in an “overall” sense, by not deviating much from \( f \) at any point in \([a, b]\). This requires that we define an appropriate notion of “distance” between functions that is, intuitively, consistent with our understanding of distance between numbers or points in space.

To that end, let \( V \) be a vector space over the field of real numbers \( \mathbb{R} \). A norm on \( V \) is a function \( \| \cdot \| : V \to \mathbb{R} \) that has the following properties:

1. \( \| f \| \geq 0 \) for all \( f \in V \), and \( \| f \| = 0 \) if and only if \( f \) is the zero vector of \( V \).
2. \( \| cf \| = |c| \| f \| \) for any vector \( f \in V \) and any scalar \( c \in \mathbb{R} \).
3. \( \| f + g \| \leq \| f \| + \| g \| \) for all \( f, g \in V \).

The last property is known as the triangle inequality. A vector space \( V \), together with a norm \( \| \cdot \| \), is called a normed vector space or normed linear space.

**Example** The space \( C[a, b] \) of functions that are continuous on the interval \([a, b]\) is a normed vector space with the norm

\[
\| f \|_\infty = \max_{a \leq x \leq b} |f(x)|,
\]

known as the \( \infty \)-norm or maximum norm. □

**Example** The space \( C[a, b] \) can be equipped with a different norm, such as

\[
\| f \|_2 = \left( \int_a^b |f(x)|^2 w(x) \, dx \right)^{1/2},
\]
where the weight function \( w(x) \) is positive and integrable on \((a, b)\). It is allowed to be singular at the endpoints, as will be seen in certain examples. This norm is called the 2-norm or weighted 2-norm. □

The 2-norm and \(\infty\)-norm are related as follows:

\[
\|f\|_2 \leq W \|f\|_\infty, \quad W = \|1\|_2.
\]

However, unlike the \(\infty\)-norm and 2-norm defined for the vector space \(\mathbb{R}^n\), these norms are not equivalent in the sense that a function that has a small 2-norm necessarily has a small \(\infty\)-norm. In fact, given any \(\epsilon > 0\), no matter how small, and any \(M > 0\), no matter how large, there exists a function \(f \in C[a,b] \) such that

\[
\|f\|_2 < \epsilon, \quad \|f\|_\infty > M.
\]

**Best Approximation in the \(\infty\)-norm**

We now consider the problem of approximating a function \(f \in C[a,b] \) by a polynomial \(p\) such that \(\|f - p\|_\infty\) is small. Such an approximation does exist; in fact, for any \(\epsilon > 0\), no matter how small, there exists a polynomial \(p\) such that

\[
\|f - p\|_\infty \leq \epsilon.
\]

This classical result can be proved by considering the interval \([0, 1]\) and using the approximation

\[
p_n(x) = \sum_{k=0}^{n} p_{nk}(x) f(k/n), \quad x \in [0, 1],
\]

where the polynomials

\[
p_{nk}(x) = \binom{n}{k} x^k (1 - x)^{n-k}
\]

are known as the *Bernstein polynomials*. It can be shown that for any given error tolerance \(\epsilon\), there exists a degree \(n\), dependent on \(\epsilon\), such that \(\|f - p_n\|_\infty \leq \epsilon\).

We now fix the degree \(n\) and consider the problem of approximating \(f \in C[a,b] \) by a polynomial \(p_n \in P_n\), where \(P_n\) is the space of polynomials on \([a, b]\) of degree at most \(n\), such that \(\|f - p_n\|_\infty\) is minimized. In fact, it can be shown that \(\|f - p_n\|_\infty\), as a function of the \(n+1\) coefficients of \(p_n\), does have a minimum on \(\mathbb{R}^{n+1}\) that can be attained, as it is a continuous function of the coefficients, and there exists a compact, nonempty subset of \(\mathbb{R}^{n+1}\) such that \(\|f - p_n\| \leq \|f\|_\infty + 1\) for any polynomial \(p_n\) whose coefficients lie in this subset. Therefore, this subset must contain a minimum.

A polynomial \(p_n \in P_n\) that minimizes \(\|f - p_n\|_\infty\), the maximum absolute value of \(f(x) - p_n(x)\) on \([a, b]\), is called the *minimax polynomial*. It is, in the \(\infty\)-norm sense, the best approximation of \(f\) on \([a, b]\) by a polynomial of degree \(n\). This polynomial, as we will see later, is in fact unique.
Example Let $f \in C[a, b]$. Then $f$ has a minimum at a point $\xi \in [a, b]$, and a maximum at $\eta \in [a, b]$. Then the minimax polynomial of $f$ of degree 0 is the constant function

$$p_0(x) = \frac{1}{2}[f(\xi) + f(\eta)].$$

We note that the error $|f(x) - p_0(x)|$ is maximized at two points, at $x = \xi$ and $x = \eta$. We also note that at $x = \xi$,

$$f(x) - p_0(x) = f(\xi) - \frac{1}{2}[f(\xi) + f(\eta)] = \frac{1}{2}[f(\xi) - f(\eta)] < 0,$$

while at $x = \eta$,

$$f(x) - p_0(x) = f(\eta) - \frac{1}{2}[f(\xi) + f(\eta)] = \frac{1}{2}[f(\eta) - f(\xi)] > 0.$$

Not only are the errors at these points of opposite sign; they are also equal in magnitude to the error on the entire interval, $\|f - p_0\|_\infty$.

A similar result holds for higher-degree approximations. The Oscillation Theorem states that if $p_n \in \mathcal{P}_n$ is the minimax polynomial of degree $n$ for $f \in C[a, b]$, then there exist $n + 2$ points $a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b$ such that

$$|f(x_i) - p_n(x_i)| = \|f - p_n\|_\infty, \quad i = 0, 1, \ldots, n + 1,$$

and

$$f(x_i) - p_n(x_i) = -(f(x_{i+1}) - p_n(x_{i+1})), \quad i = 0, 1, \ldots, n.$$

These points are called the critical points of $f$ on $[a, b]$.

This result can be used to prove the uniqueness of the minimax polynomial. It can also be used to compute it. It follows from the Oscillation Theorem that if $f \in C[a, b]$ is continuously differentiable on $(a, b)$, and if $f'$ is monotonic on $(a, b)$, then the minimax polynomial of degree one, $p_1(x) = c_0 + c_1x$, can be obtained by first noting that because $f'$ does not change sign on $[a, b]$, $f - p_1$ must assume its maximum and minimum values at $x = a$, $x = b$, and $x = d$, where $d \in (a, b)$.

Because $p_1$ is a minimax polynomial, it follows that

$$f(a) - (c_0 + c_1a) = A,$$

$$f(d) - (c_0 + c_1d) = A,$$

$$f(b) - (c_0 + c_1b) = A.$$

It follows from the first and third equations that

$$c_1 = \frac{f(b) - f(a)}{b - a}.$$
That is, the graph of $p_1(x)$ is parallel to the secant line passing through $(a, f(a))$ and $(b, f(b))$.

From the first and second equations, we obtain

$$c_0 = \frac{1}{2} [f(a) + f(d) - c_1(a + d)],$$

where $d$ is the point at which the slope of the tangent line is equal to $c_1$, the existence of which is guaranteed by the Mean Value Theorem. This point is also unique, because $f'$ is assumed to be monotonic. We conclude that $p_1$ is the linear function whose graph is a line that is parallel to the secant line and the tangent line at $d$, both of which have slope $c_1$, and is halfway between these lines.