Determinants in Wonderland

The Rev. Charles Lutwidge Dodgson, better known as the author of “Alice in Wonderland,” was also a mathematician. He developed an easy, elegant method to compute determinants of matrices. Unfortunately, the method often fails! This poster describes a modification of Dodgson’s method that allows it to work for many more matrices—but still not all.

1 Background

Many important problems in science and engineering require us to evaluate the determinant of a matrix, such as:

\[ M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & 5 \\ 1 & 3 & 3 \\ 2 & 3 & 5 \end{pmatrix} \]

Algebra students learn to compute determinants by expanding cofactors.\[ ]

Example 1.

\[ \det M = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} = 1 \det \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + 1 \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 2. \]

2 The problem

Dodgson’s method works only if there are no zeroes in the interior of \( M \), and if there are no zeroes in the interior of any of the \( M_i \)’s generated by the method.


\[ M = \begin{pmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & -2 \\ 1 & 1 & -2 & -1 & -1 \\ 2 & 1 & 1 & -2 & -1 \\ 1 & 1 & -2 & -1 & -2 \end{pmatrix} \]

has no zeroes in its interior, but when \( i = 3 \) we obtain the matrix

\[ M_i = \begin{pmatrix} -15 & 6 & 12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{pmatrix} \]

The zero in the interior of this matrix makes it impossible to compute \( M_i \). ♦

Can we modify Dodgson’s method to work around this?

3 Analysis

Dodgson’s method is based on a result of Jacobi [2].

Theorem 2 (Jacobi, 1833). Let

- \( M \) be an \( n \times n \) matrix;
- \( A \) be an \( m \times m \) minor of \( M \), where \( m < n \);
- \( A' \) be the corresponding minor of the adjugate of \( M \); and
- \( A'' \) the complementary \((n-m) \times (n-m)\) minor of \( M \).

Then

\[ \det A'' = (\det A')^{-1} \cdot \det A'. \]

How does this theorem give us Dodgson’s method?

4 A modified Dodgson’s method

If a zero appears in the interior, we can adapt the method. Simply choose \( A' \) from a different row and/or column, crossing them out to obtain the minor \( A \).

Example 4. This matrix fails at \( i = 2 \) using the original Dodgson’s method:

\[ M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \]

The modified Dodgson’s method works perfectly:

\[ M_i = \begin{pmatrix} -2 & -1 & 1 \\ 4 & -2 & 1 \end{pmatrix} \]

\[ M_j = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix} \]

\[ M_j = (1) \]

Thus \( \det M_j = 1 \).

How did we compute \( M_j \)? The highlighted zeroes in the interior of \( M \) mean that to compute the element in row 1, column 2 of \( M_j \), we have to use a different minor. The elements above the highlighted zeroes are non-zero, so we apply the modification suggested above to find it. To compensate for the zero in row 2, column 3, we crossed out row 1, column 3, obtaining

\[ A' = \det M_{2,3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ A'' = \det M_{2,3} - \det M_{2,2} \]

We find each element of \( A' \) by crossing out the corresponding element of the 3x3 minor around \( M_j \) and taking the determinant. The careful observer will notice that we already computed \( M_j \) and \( A'_{1,2} \) when computing \( M_j \), \( A'_{1,2} = M_j_{1,2} \) and \( A'_{1,3} = M_j_{1,3} \).

The method likewise computes the determinant for Example 3, arriving at 36. There are two drawbacks.

1. The modified method is not nearly as easy as the original method.

2. If any iteration contains a 3 \( \times \) 3 block of zeroes, it fails. An "easy" example:

\[ \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ \det = 2 \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 0 - 1 \cdot 1 \cdot 0 = 2 \]

\[ = 2^2 = 64. \]

We have successfully implemented the new method on the computer algebra system Sage.\[ ]

References


Dear Leggett, Perry, and Sanders,

You have written a compelling and clear explanation of Dodgson’s method and its modifications. Thank you for your hard work and dedication to the field of mathematics. I am genuinely impressed by your ability to make complex mathematical concepts accessible to students and professionals alike. Your presentation not only highlights the elegance of Dodgson’s original method but also demonstrates the important role it plays in modern mathematics. Keep up the excellent work, and continue to inspire others with your passion for mathematics.

Sincerely,

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