Appendix ACT

Activities

This document contains classroom activities for single variable calculus and pre-calculus. Activities are provided in the order in which topics appear in the text with one notable deviation. Because of typesetting constraints, the early activities are those that have a wide margin as in the main text. These are followed by activities that are set with small margins.

ACT.1 Purpose of the Activities

The best answer to the question, "What is the most effective method of teaching?" is that it depends on the goal, the student, the content, and the teacher. But the next best answer is, "Students teaching other students."
(Wilbert McKeachie)

Facilitating active learning is a challenging task. There is no one method that guarantees to engage all students and keep them active. This is because students' preferred learning styles can differ widely even in a class as small as 20 students. For a classification of various learning styles see the Learning Styles Inventory at `http://www.engr.ncsu.edu/learningstyles/ilsweb.html`.

On the positive side, although there is no one method guaranteed to keep all students engaged, there also is no one method that turns all students off. Advocacy of active learning is sometimes mistaken as "lecture bashing." This is either a misperception or, if lecturing actually is belittled, highly inappropriate. For example, lecturing and verbal communication are a perfect fit for verbal learners. Moreover, lecturing is very effective for transmitting the same information to a large group of people.

The activities presented here are designed to make lecturing more effective by keeping students engaged or preparing them for the next segment of a lecture.

When placed appropriately, they can enhance the value of a lecture. Depending on the context, the author has conducted successful class periods that ranged from 100% lecture to 10-20% lecture with the average being around 80-90% lecture.[1] The descriptions presented here are intended as examples how the author believes the activities can be used. It is virtually certain that there are other ways that will fit better with other individuals' teaching styles. Please consider this document as food for thought to devise your own approaches. The main goal is to keep students' minds engaged throughout the class period. This is what active learning means – it is not tied to a particular style of instruction. Another goal is to improve students' communication ability, because it is virtually assured that they will work in teams as professionals. Here is where cooperative learning is important, because communication with each other seeds the abilities needed later as professionals.

ACT.2 Types of Activities

Overall, the activities can be classified into two broad, overlapping classes.

Guided Discovery. Some activities give a set of instructions that can be carried out without initial explanation. After this is done, students are asked to identify an underlying pattern. For example, the "Discovering Substitution" part of the activity on page ACT-29 can be used at the beginning of class to introduce the idea behind substitution. Not every student will answer the question "Why does this work?" with a theorem describing substitution. However, students are put in the right mind-set by the discovery activity and any subsequent explanation/proof will go more smoothly. Guided discovery can be done individually or as a "brainstorm" type group work. The author usually lends assistance to students and asks leading questions during the activity. Conceptions and misconceptions observed during this phase guide the presentation of the underlying theorem, the proof, proof sketch or plausibility argument and the first few examples.

Practice. Any fundamental technique requires a certain amount of practice. Practice activities require the student to apply a new technique in a number of contexts. Essentially, the students work examples that the instructor would otherwise lecture about.[2] The "Working with Substitution" part of the activity on page ACT-29 is a good example. Usually, after presenting one or two examples, the author lets students practice. Practice activities can also be implemented with the active/coorperative learning techniques in the next section. Again, the author also lends assistance as he observes students work. After students have made a certain amount of progress, the author starts writing solutions on the board. Students can choose to compare their work with the author's, copy notes in case they did not get far, or ignore the author, if they are further along. Once solutions are on the board, the author briefly explains steps, putting specific emphasis on how to avoid mistakes that he saw as he worked with students.

ACT.3 Some Active/Cooperative Learning Techniques

This section must be started with a warning. Anyone who tries to implement all of these techniques at once most likely will not meet with success. Adjusting one's

[1] Rich Felder, a convincing and strong advocate of active learning, said that he devotes about 15% of his class time to active learning activities.

[2] Jenna Carpenter once said that students get more out of one or two examples that they work themselves while she observes and lends assistance (even if they don't complete them all the way), than out of four examples that she could write on the board during the same time. The author agrees.

teaching style is a lengthy process and there is no one sure path to success. The standard recommendation is to try one or two techniques in a term and decide if they work or not. If they do, continue their use. If not, check if a modification will do, or abandon the technique for now.

- **Think-Pair-Share.** Two partners solve the same problem independently. After they are done, they compare methods and results. Differences are explained and, if they stem from mistakes, eliminated.

  Trains oral communication, the justification of steps that differ from someone else's, the tracking of mistakes and the ability to accept different solution methods.

- **Peer Editing.** Similar to Think-Pair-Share, two partners solve the same problem independently. After they are done, they trade their solutions and edit/grade the partner's work.

  Trains written communication and the spotting of mistakes. Also shows that if peers have trouble following a solution, then the instructor most likely will, too.

- **Thinking Aloud Paired Problem Solving (TAPPS).** One partner is the "talker", the other is the "listener" in this activity. The talker's job is to solve a problem as perfectly as possible, while explaining what is done in every step. Essentially the talker gives a mini-lecture to the listener. The listener's job is to listen and spot mistakes. As mistakes occur or the talker gets stuck, the listener is *not* supposed to take over, even if the listener knows the solution. Instead, the listener is supposed to ask leading questions such as "Are you sure your addition is right?" or "Why don't you try rationalizing the denominator?" to put the talker back on the right track.

  Roles should be reversed regularly to give both partners the talker and the listener experience.

  Trains oral communication and listening skills. Based on the insight that the highest rates of retention are achieved in tasks in which one does something while explaining it (for some details, see R. B. Lewis (1991), *Creative Teaching and Learning in a Statics Class*, Engineering Education, 15-19).

- **Jigsaw.** This activity works for groups of various sizes. Each group member becomes a "specialist" who is assigned a different task to complete. This can be done alone, in a full jigsaw, by joining the other groups' specialists with the same task. When forming groups of specialists with the same task, usually some training is involved before the task can be completed.

  Upon completion of tasks, in a full jigsaw the original groups are re-formed, otherwise, students just agree to start reporting, and each group member explains his/her task to the remaining group members.

  Trains oral communication and listening skills. To enforce individual accountability for all parts of the jigsaw, a test could include questions on each task.

- **Enhanced lecture.** This idea only involves the instructor. Stop for a minute or two halfway through a longer lecture to allow students to re-focus. The author has also heard of colleagues make students do jumping jacks when they note that attention is wavering.

> It sounds a bit silly and it is hard to keep yourself silent for a whole minute with nothing to do, but it re-focuses the class. It is important to make sure that during such a short break people do not reach for distractions. We are somewhat conditioned to think that we always have to do something. A quick peek at a newspaper during a short break would negate the break's effect. On the other hand, going to the bathroom should be o.k.

Combinations of the above techniques can also be effective. For example, after a certain time during an activity, students find their own way to enhance the lecture. Sometimes they are stuck and the break is forced, sometimes they are done, sometimes they simply need a break. Also, TAPPS activities sometimes turn into group problem solving if both partners get stuck. There is nothing wrong with such modifications as long as students remain actively engaged with the material.

Aside from students actively engaging the material, a big benefit of these techniques is that the instructor gets to observe students at work. These observations as well as questions asked by students in individual discussions can be used to guide further presentations. Experienced instructors can often predict what parts of a class will be the most troublesome for students. If the instructor teaches a class for the first time, the feedback through observations can make up for lack of experience. Moreover, even in classes the author has taught frequently, observations can reveal student difficulties that he was previously unaware of.

Shifting, Stretching and Reflecting

Throughout this activity we shall consider the function $f(x) = x^2$.

1. Graph $f(x) = x^2$ on a CAS.

2. Graph the functions $g(x) = f(x) + 3$ and $h(x) = f(x) - 2$ in the same window as f. Describe how these graphs differ from the graph of f. Is there a general rule you might recognize? Explain why the rule should work for all functions.

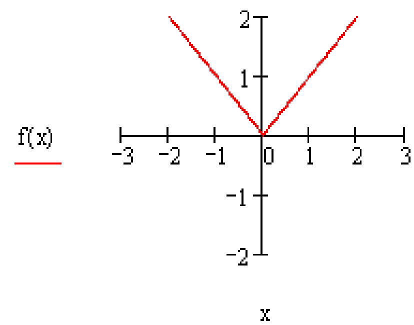
3. Graph the functions $g(x) = f(x + 4)$ and $h(x) = f(x - 1)$ in the same window as f. Describe how these graphs differ from the graph of f. Is there a general rule you might recognize? Explain why the rule should work for all functions.

4. Graph the functions $g(x) = 2f(x)$ and $h(x) = \frac{1}{3}f(x)$ in the same window as f. Describe how these graphs differ from the graph of f. Is there a general rule you might recognize? Explain why the rule should work for all functions.

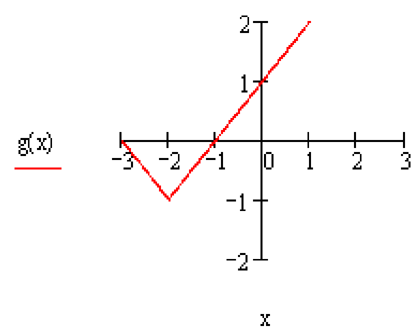
5. Graph the function $g(x) = -f(x)$ in the same window as f. Describe how this graph differs from the graph of f. Is there a general rule you might recognize? Explain why the rule should work for all functions.

6. For the last exercise graph the functions $f(x) = x^3 - 2x^2$ and $g(x) = f(-x)$ in the same window as f. Describe how this graph differs from the graph of f. Is there a general rule you might recognize? Explain why the rule should work for all functions. Also explain why we did not use the function $f(x) = x^2$ or the function $f(x) = x^3$ for this exercise.

After the discovery activities 1 to 6 consider the following two applications.

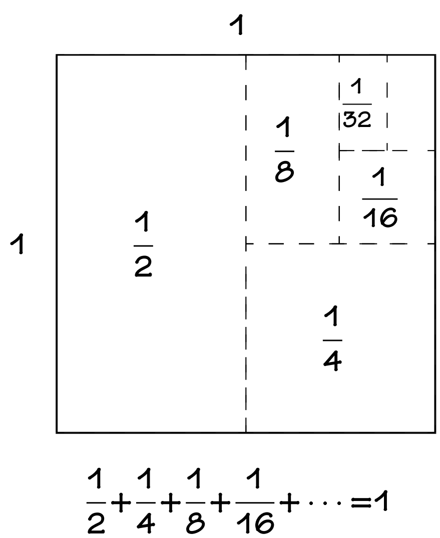


With the graph of f given as above, find the graph of $f(x + 1) + 2$.



Write the function g above as $g(x) = f(x - a) + b$, where f is as in the top graph.

Series of Numbers



(Visualization.)

Series with a finite sum.

Consider the following hypothetical situation faced in a hospital.

- A patient receives injections with $\frac{1}{2}$g of a medication every 24 hours.
- Half the medication that is in the patient's system is broken down and excreted within a 24 hour period, leaving the remaining half of the medication still in the patient's system.

The question now is how the medication accumulates in the patient's body over the long term. Knowing the answer to this question prevents accidental overdoses. Let us assume that levels of up to 10g of the medication in a patient's body are considered uncritical. With the above indicated intake and breakdown of the medication, will the patient overdose?

1. Compute the amount of medication in the patient's body right after the first injection (a_1), after the second injection (a_2), after the third (a_3), etc.

2. Find a formula for the amount of medication in the patient's body after k injections (a_k).

3. Compute the limit of the a_k as $k \to \infty$. This is the long term accumulation of medication in the patient's body. Is this level a dangerous level?

Unbounded series.

Not every series has a finite sum, as is easily seen with the series $\sum_{n=1}^{\infty} n$. Some series however are tempting the investigator to hope they are finite, because their partial sums grow so slowly. One such series is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. In the following we will show that the harmonic series grows beyond all bounds.

1. Compute the sum of the first term, of the next two terms, of the next four terms, of the next eight terms, etc.

2. In general, write an expression for the sum of the 2^l-th term through the $(2^{l+1}-1)$-st term. How many terms does the sum have? Find a lower bound for all terms of this sum.

3. Use your estimates above to show that the sum of the 2^l-th term through the $(2^{l+1}-1)$-st term is at least $\frac{1}{2}$.

4. Conclude that the harmonic series cannot have a finite sum.

Differentiating Products

It is very tempting to *hope* that $\frac{\mathrm{d}}{\mathrm{d}x}(fg) \stackrel{?}{=} \left(\frac{\mathrm{d}}{\mathrm{d}x}f\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}g\right)$. Consider the functions $f(x) = 3x^2 + 1$ and $g(x) = x + 2$. Find

1. $(fg)(x) =$

2. $\frac{\mathrm{d}}{\mathrm{d}x}(fg) =$

3. $\left(\frac{\mathrm{d}}{\mathrm{d}x}f\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}g\right) =$

In light of this data, is the above hope for a simple product rule still worth considering? Let us use difference quotients to find a rule to differentiate *general* products $F(x) = f(x)g(x)$. We know the difference quotients for f and g converge to the respective derivatives for $h \to 0$. Hence these difference quotients are manageable patterns that we will search for in the following computations. (Use general f, g, not the functions above.)

$(fg)'(x) = \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h}$ — Use the definition of the derivative.

$=$ — Simplify: resolve fg.

$=$ — Add zero in in the numerator in the form -f(x)g(x+h)+f(x)g(x+h).

$=$ — Regroup the terms in the numerator, identify terms of the form f(x+h)-f(x) and g(x+h)-g(x).

$=$ — Factor out like terms.

$=$ — f(x+h)-f(x) and g(x+h)-g(x) are the ingredients for f'(x) and g'(x). Separate the expression to identify the difference quotients.

$=$ — Since all involved limits do exist, we can move the limit into the summands resp. factors.

$=$ — This is now easily identified as

Activity: Organizing Measured Data

In this activity we consider some basic tools to analyze data. For this activity, we use data that has been generated using random number generators. Open any of the data files supplied at `http://www2.latech.edu/~schroder/sprsh.html` in a spreadsheet. The data is recorded in column B, starting in cell B101, under the heading "input". This is input data for our analysis. Column A contains a running count, starting in cell A101, under the heading "count". There should be at least 200 data points which are all numbers in the interval [0, 11).

A natural idea for analyzing data is to find the **average**. That is, if n data points $x_1, \ldots, x_n$ are given, we want to find $\overline{x} := \frac{1}{n}\sum_{k=1}^{n} x_k$. The average tells where the values are "centered". However it does not tell if the data clusters closely around the center or if it is spread wisely.

As a measure of spread we measure the average distance of the data points from the center. A quantity that measures this distance is $\frac{1}{n}\sum_{k=1}^{n}|x_k - \overline{x}|$. This quantity is not very widely used. The quantity used more widely is the **(sample) standard deviation** which is defined to be $s := \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(x_k - \overline{x})^2}$. Squaring takes care of the sign of the distance to the center. Taking the square root after summing scales the result back from a squared scale to the regular scale of values. (Consider the units for further motivation.) The reason why we divide by $n-1$ rather than by n when we average the values will be explained in Section 18.1.

- **Task 1.** Compute the average and the standard deviation for your data set.

These two numbers give some information about the values we are analyzing, but that information is quite coarse. To obtain more information about the way the values are distributed we count how many values are below 1, below 2, 3, ..., 11.

- **Task 2.** Use the spreadsheet to count the values below $1, 2, 3, \ldots, 11$.

Now that we have generated the data that tells us how many values are below $1, 2, 3, \ldots, 11$, we can determine how many values are between any two integers between 0 and 11. The number of values that are between a and b is the number of values below b minus the number of values below a.

- **Task 3.** Generate a list which tells how many values are in the intervals $[0, 1), [1, 2), \ldots, [10, 11)$. (This list is called a **frequency list**.)

Having lists of numbers helps, but often graphical information is more helpful in the analysis of data.

- **Task 4.** Generate bar charts in which the height of each column represents the number in the lists generated in Tasks 2 (also called a cumulative distribution chart) and 3 (also called a **histogram**).

Things to report/questions to answer.

- Print out the charts generated in Task 4 (or sketch them).
- Report the average and the sample standard deviation of your data set.

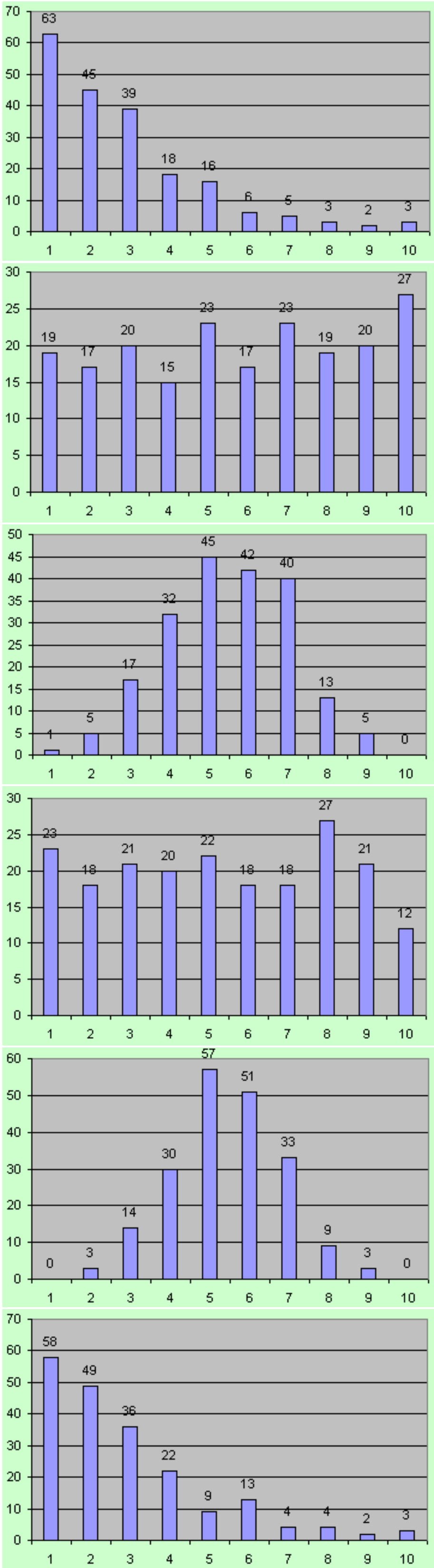


Figure ACT.1: Aside from practicing how to organize data, the present activity also shows that randomness comes in different shapes. The above six histograms come from three different random processes (two from each process). Identify the pairs of histograms that are from the same process.

- Verbally describe the shape of your frequency distribution chart (the histogram). You may also compare them with shapes of functions that you know.
- Comment on how well the average and the sample standard deviation reflect the actual distribution of the values for your data set.
- Compare the charts generated by different groups. Are the distributions the same? Do they look similar or do they appear fundamentally different?
- Do you expect the shape of your frequency distribution chart to change drastically when refining your counting, say, by counting all values in the intervals $\left[0, \frac{1}{2}\right)$, $\left[\frac{1}{2}, 1\right)$, $\left[1, \frac{3}{2}\right)$, etc.?

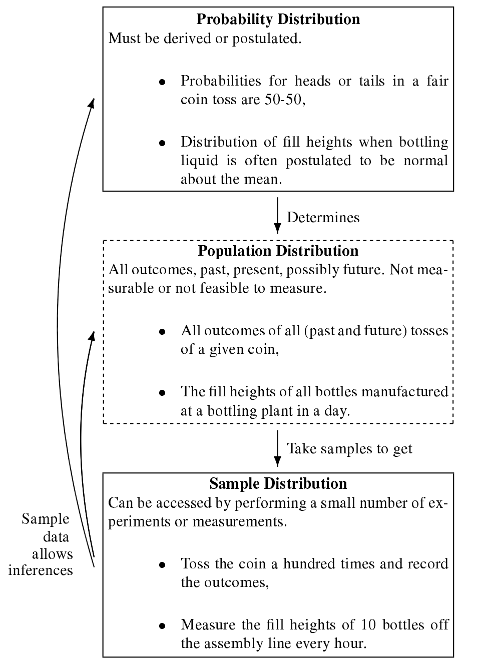


Figure ACT.2: Relation between the probability distribution, the actual population and the sample population. The downward pointing connections are normally considered probability theory, the upward connections statistics.

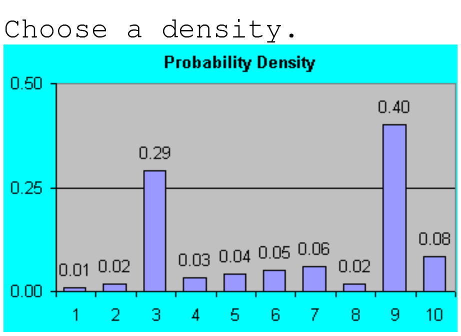
Comparing Probability Density Functions with the Populations they Generate

It is easy to say that the underlying probability density function of a certain process governs its behavior. However, it is much harder to act accordingly. For example, the author fully understands that the odds for winning the lottery are astronomically high against winning. Yet he has bought a lottery ticket or two.[3] Is it rational or irrational to bet against an intangible quantity such as the probability density function?

The relationship between the probability density function, the total population of outcomes and the samples that are accessible to an investigator is pictured in Figure ACT.2. Note that while the probability density function can be derived or postulated and small sample populations can be measured, the total population of *all outcomes ever* taken from a given random process is generally not accessible. It may be inaccessible by default. For example, we will never know all possible measurements that a human experimenter can make to find the frequency of a pendulum. The total population may also be inaccessible because it is too large. For example, the number of possible combinations that occur in a lottery is too large to allow detailed analysis of each outcome. This inaccessibility is what makes statistics a tricky subject. We will always be forced to work with hypotheses that can only be verified on the basis of partial data (if at all).

To improve our intuition, we shall model a large number of random experiments that return integers between 1 and 10. We will not perform physical experiments, but simulations of these experiments using the EXCEL tool `centlimexc95.xls`, available at `http://www2.latech.edu/~schroder/sprsh.html`.

In `centlimexc95.xls` it is possible to pick the probability distribution for experiments that have outcomes among the numbers 1 through 10.[4]



1. Start the workbook `centlimexc95`.

2. Go to the worksheet "Probability to Population".

3. Some densities are provided in the workbook and can be activated by selecting them. Densities that have parameters will show which parameters can be adjusted. Try them.

4. If you don't like the "canned" densities, you can use the sliders below the picture to adjust the probabilities to your liking. As this is done, you determine the likelihood of a random experiment with this density to return a certain numerical value between 1 and 10. Try out this process, too.

5. For each density that you like,

   - Set the population size equal to 1 and record where (in repeated trials) most outcomes are likely to be,
   - Once that gets too tedious, take a larger sample population and look at its distribution.
   - Compare the shape of the population distribution with the shape of the probability density for different sizes of the population.

[3] In my defense I have to say that it was never more than 2. To put it in a nutshell, if the jackpot was at $150,000,000 and I had $100 to "invest", I might buy a ticket or two and spend the rest of the money on a nice dinner with my family.

[4] The tool does not allow continuous random variables. However, we could assume values have come from continuous random variables and the have been combined in bins that contain all outcomes between 0 and 1 (bin 1), 1 and 2 (bin 2), etc.

This is the electronic version of “simulated rolls of the dice”. The first experiment is the rolling of one die, the second experiment is the rolling of multiple dice, with the computer returning the sorted results.

6. Say “Oooohh, Aaaahh, this is soo cool.” (Just checking who is awake.)

7. Decide on one probability distribution that you want to investigate further. **You will work with your chosen distribution exclusively for the rest of this activity and also in the activities on pages ACT-9 and ??, so please record your parameters or your values.** Record the 10 numbers (the probabilities of the “bins”) that characterize the probability distribution below.

   Probability, bins 1-5

   Probability, bins 6-10

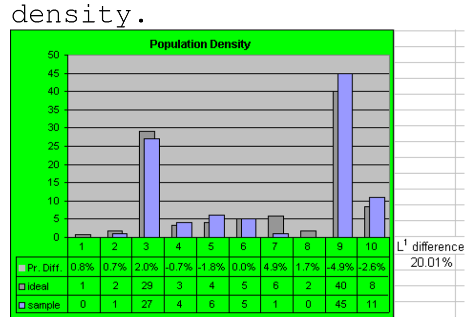
8. Start with a small population size and slowly increase the population size. What happens to the shape of the population distribution? For populations of size 10, 50 and 100 find the % difference between population and probability distribution as recorded on the spreadsheet (called “L^1-difference” there) for 8 consecutive populations each.

   Size 10:

   Size 50:

   Size 100:

Record the L^1-differences between population and density.



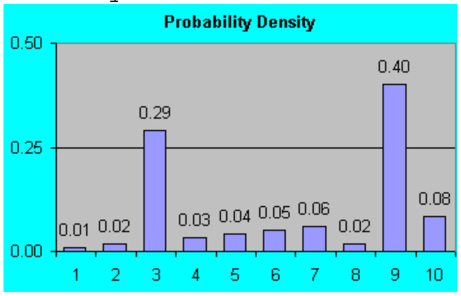
Answer the following questions.

- For each population size, what is your average %-difference?
- Did the sample populations you generated match the idealized population?
- Did your sample populations get closer to the idealized population as you increased the population size?
- For single measurements (population size 1) do you have any way of predicting before the experiment what the outcome will be?
- For large populations (say of size 100) do you have any way of predicting how the measured population will be distributed?

Probability Density Functions and Cumulative Distribution Functions

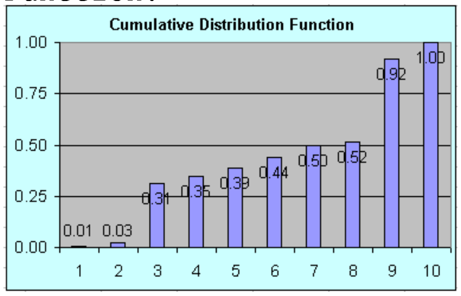
In this activity we will ultimately design a density function that has a certain cumulative distribution function.

Density.



1. Start the workbook `centlimexc95`.
2. Go to the worksheet "Density and CDF".
3. Just as before you can choose which density you want to investigate. This time the right graph shows the corresponding cumulative distribution function $P(X \leq n)$, where n is an integer between 1 and 10. Note that since everything is discrete we will add instead of integrate. We have 10 events, the individual probabilities of which add up to 1.
4. Answer the following questions.
   (a) What happens to the shape of the cumulative distribution function when the probability density function assigns most of the probability to a small range of adjacent values (say 2 or 3 adjacent values)?
   (b) What happens to the shape of the cumulative distribution function when the probability density function assigns roughly equal probability to all values?

Cumulative Distribution Function.



5. Please generate a density that has the function

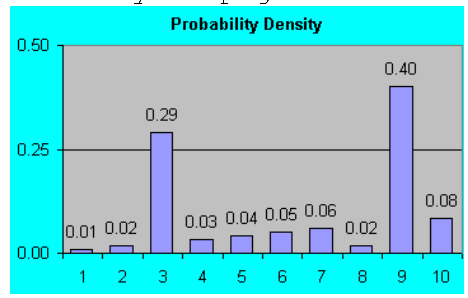
n	1	2	3	4	5	6	7	8	9	10
$P(X \leq n)$	.1	.15	.21	.34	.60	.60	.65	.72	.75	1

as its cumulative distribution function. (This can be attempted with the sliders or through direct input of values for the density.)

Sample Averages of Discrete Densities

In this activity we investigate how the averages of samples from a given population behave as we vary the sample size. Experimenters normally only have effective access to small samples. This is why we are interested in the averages of small samples.

Recall your density from activity on page ACT-6.



1. Start the workbook `centlimexc95`.

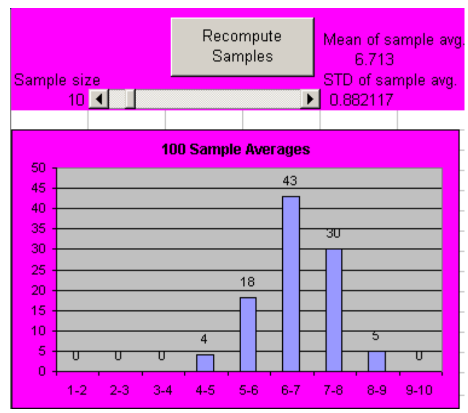
2. Enter the density that you investigated in the activity on page ACT-6 as your custom density and generate a population of 1000 samples. Then go to the sheet "Sample averages".

Then compute a sample population of size 1000.

3. Our population of 1000 now serves as our underlying population and we will take 100 smaller samples (with replacement) out of this population and compute their averages. The size of the smaller samples *must* be adjusted with the slider. This size is accessed from two sheets ("Sample averages" and "Sample STDs") and the cell that stores the value that is used is on a different sheet. To see the distributions of the individual small samples, scroll down to the light green histogram.

For increasing sample sizes compute the distribution of the sample averages.

4. For the following tasks compute the sample averages for sample sizes of 5, 10, 20, 40, 80 and 100



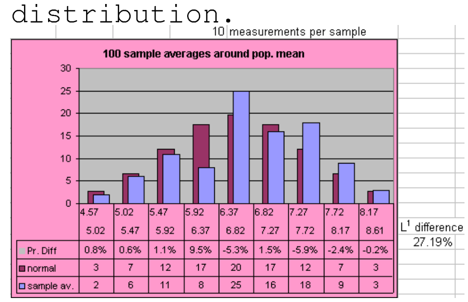
5. First consider the left (purple) histogram and describe what (on the 1-10 scale) happens to the sample averages as the sample size increases.

   What does the mean of the sample averages approximate as you increase the sample size? (We are considering an average of 100 averages here.)

   (Note: As the sample size is increased, more values are added to the existing samples. Existing values are not re-sampled unless you click "Re-compute Samples".)

6. The sample averages themselves are scattered and their spread can be measured by computing the standard deviation of the sample averages. What happens to the standard deviation of the sample averages as you increase the sample size? Does doubling the sample size cut the standard deviation in half?

Now consider the shape of the distribution compared to the "right" normal distribution.



7. Now consider the right (pink) histogram and note what happens to the averages of the small samples as the sample size increases. The scale for this histogram smaller and it is centered around the mean of the underlying population. Start with a bin size of 1 and pay particular attention to the *shape* of the distribution. Describe the shape of the distribution.

   What happens as you shrink the bin size?

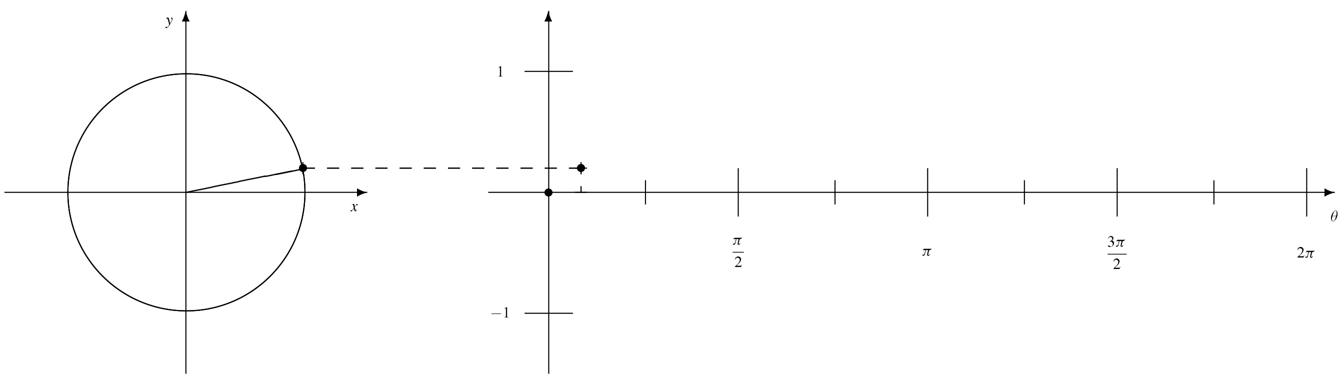
Finally note that if the bin widths are chosen too small, then we will not see bell-shaped curves that are almost equal to each other. Rather, we will see a set of bins, some of which are empty and some of which contain a small number of sample averages. No structure is visible. This is an indication that the a resolution is too high for the data that we have. Note that this kind of phenomenon is an intrinsic limitation of discrete and experimental approaches. Only the use of observed data in conjunction with the continuous approach made it possible to discover a tool as powerful as the central limit theorem.

The Graphs of the Sine and Cosine Functions

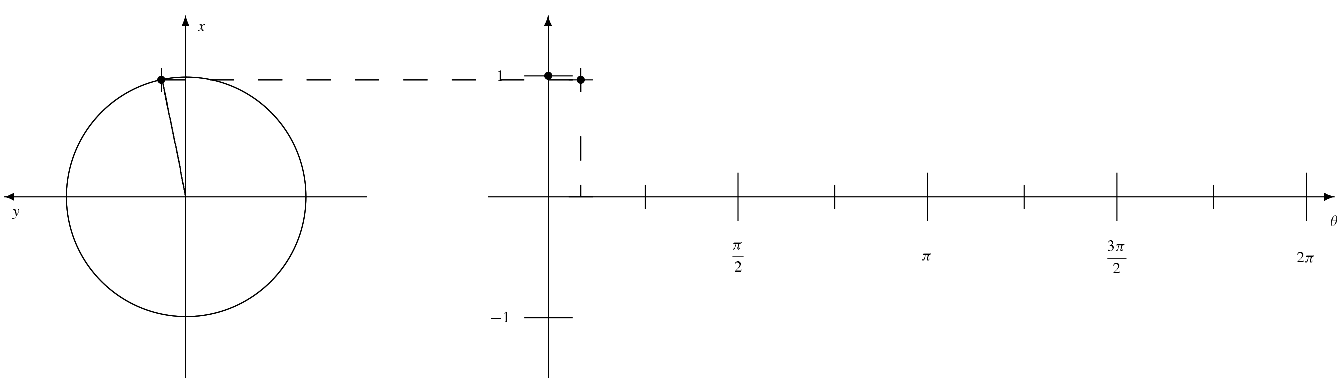
In this activity you will sketch a full period ($0°$ to $360°$ or 0 to 2π) of the graphs of the sine and cosine functions. Instructions are essentially the same for both functions.

1. In the given circle, mark all angles that are multiples of $15° = \dfrac{\pi}{12}$.
2. For each angle, translate the y-coordinate (for the cosine function the x-coordinate) of the intersection point of the terminal side of the angle to the appropriate point in the graph.
3. After all points have been plotted, sketch a smooth curve through all points. This is the graph of $\sin(x)$ ($\cos(x)$) on $[0°, 360°]$ or $[0, 2\pi]$, respectively.

Sine function.



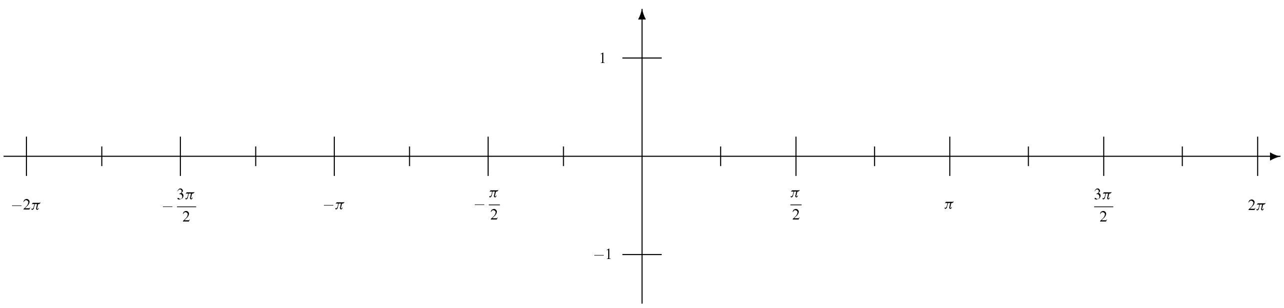
Cosine function.

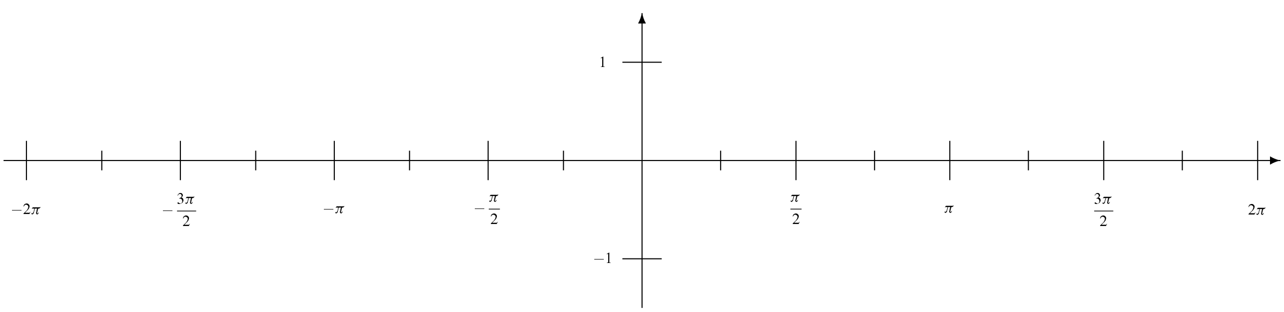


Discovering Some Trigonometric Identities

A negative angle identity.

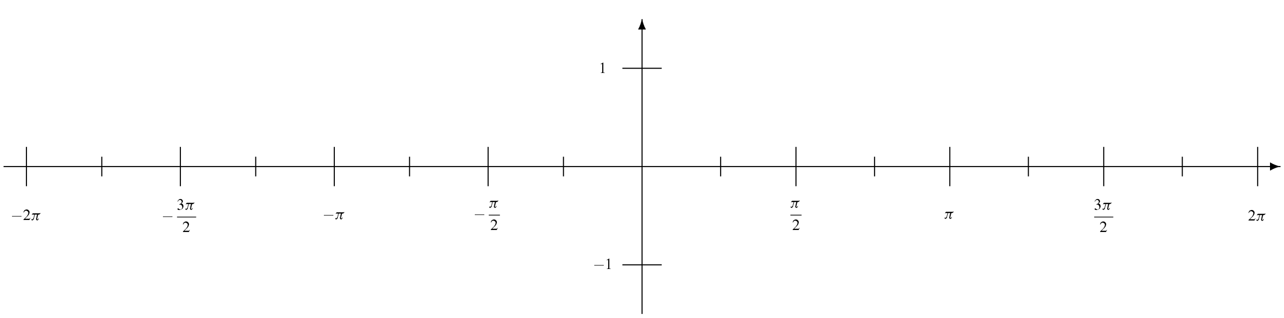
1. Sketch the graph of $f(x) = \sin(x)$ on the interval $[-2\pi, 2\pi]$. Use the fact that the sine function has period 2π.
2. Underneath the graph of the sine function, sketch the graph of $g(x) = \sin(-x)$. (Essentially you are reflecting the graph of $\sin(x)$ across the y-axis.)
3. Examine the two graphs and obtain an equation that relates $\sin(x)$ to $\sin(-x)$.





A cofunction identity.

1. Use the above graph of the sine function to obtain the graph of $h(x) = \sin\left(\frac{\pi}{2} - x\right)$. (Essentially you are reflecting the graph over the line $x = \frac{\pi}{2}$ and then re-centering the origin.)
2. Examine the graph and obtain an equation that relates $h(x) = \sin\left(\frac{\pi}{2} - x\right)$ to another trigonometric function.



Amplitudes, Frequencies and Phase Shifts

CAS for sketches allowed.

Discovery.

1. Sketch the graphs of $f(x) = \sin(x)$ and $g(x) = 5\sin(x)$ in the same coordinate system. What are the differences and similarities between the graphs? How does the graph of $g(x)$ change if the 5 is replaced with a $\frac{1}{3}$?

2. Sketch the graphs of $f(x) = \sin(x)$ and $g(x) = \sin(3x)$ in the same coordinate system. What are the differences and similarities between the graphs? How does the graph of $g(x)$ change if the 3 is replaced with a $\frac{1}{4}$?

3. Sketch the graphs of $f(x) = \sin(x)$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$ in the same coordinate system. What are the differences and similarities between the graphs? How does the graph of $g(x)$ change if the $-\frac{\pi}{3}$ is replaced with a $+\frac{\pi}{4}$?

Application.

Sketch the graph of the function
$f(x) = 3\cos(5x + 2)$.

Systems of Linear Equations

1. Determine if neither, one or both of the given tuples of numbers solve the system of equations.

(a) $(4, -1, 1)$, $(8, 1, 3)$

$$\begin{array}{rcrcrcr} -x & + & 2y & + & 3z & = & 3 \\ x & + & 2y & - & 4z & = & -2 \\ 2x & + & 3y & + & z & = & 7 \end{array}$$

(b) $(-1, -1, 1)$, $\left(-\frac{1}{6}, \frac{1}{4}, \frac{7}{12}\right)$

$$\begin{array}{rcrcrcr} -x & + & 2y & + & 4z & = & 3 \\ x & + & 2y & - & 4z & = & -2 \\ 2x & - & y & + & z & = & 0 \end{array}$$

(c) $(1, 2, 3)$, $(-4, 1, -4)$

$$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & -7 \\ -x & - & 2y & + & z & = & -2 \\ & - & 7y & + & z & = & -11 \end{array}$$

2. Solve the system of equations using the elimination method.

(a)

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 4 \\ 2x & - & y & + & z & = & -2 \\ 2x & - & y & + & 2z & = & 0 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 3x & - & y & + & 2z & = & 4 \\ 2x & + & 4y & + & z & = & -2 \\ 2x & - & y & + & 2z & = & 0 \end{array}$$

(c)

$$\begin{array}{rcrcrcr} -2x & + & 3y & - & z & = & -1 \\ x & + & 2y & - & 2z & = & 4 \\ -4x & + & 13y & - & 7z & = & 0 \end{array}$$

(d)

$$\begin{array}{rcrcrcr} 4x & - & 2y & - & 3z & = & 2 \\ 2x & + & y & - & 5z & = & 1 \\ -2x & + & 7y & - & 9z & = & 1 \end{array}$$

3. Find the equation of the quadratic function that goes through the given points.

(a) $(1, 1)$, $(2, 6)$, $(-1, 9)$,

(b) $(1, 2)$, $(3, 8)$, $(4, 11)$.

4. Two liters of a 35% alcohol solution are to be made by mixing 60%, 40% and 15% solutions. The amount of 40% solution is supposed to be 3 times the amount used of the 60% solution. How much do we need to use of each solution?

Subsequences

1. For the sequence $\frac{1}{2}, \frac{9}{10}, \frac{1}{4}, \frac{19}{20}, \frac{1}{8}, \frac{29}{30}, \frac{1}{16}, \frac{39}{40}, \ldots$
   (a) Find the next four terms.
   (b) Determine if you think the sequence converges. Explain your reasoning.

2. For the sequence
$$\left\{\left(1+(-1)^k\right)2^k\right\}_{k=0}^{\infty}$$
   find
   (a) A subsequence that goes to infinity,
   (b) A subsequence that goes to zero.

3. Does the sequence
   $\{a_k\}_{k=0}^{\infty}$ with
$$a_k = \begin{cases} 1; & \text{for } k \neq 10^n \text{ for any } n, \\ 0; & \text{for } k = 10^n \text{ for some } n, \end{cases}$$
   converge or diverge? Explain.

4. Is it possible for a sequence to converge even though it has a subsequence that goes to plus or minus infinity? Explain.

5. Is it possible for a sequence to converge even though two subsequences have different limits? Explain.

6. In the interval [0, 1] attempt to define or draw a sequence $\{a_k\}_{k=s}^{\infty}$ so that the terms a_k do not cluster near any point.

   Then read the Bolzano-Weierstrass theorem and comment on the task.

Behavior of Functions As $x \to \infty$

1. Does $f(x) = \dfrac{3x^2 - 5x + 3}{7x - 4x^2 + 1}$ converge as $x \to \infty$? Explain.

2. For the function $f(x) = \sin(x)$

   (a) Find a sequence $a_k \to \infty$ so that $f(a_k) \to 1$.

   (b) Find a sequence $a_k \to \infty$ so that $f(a_k) \to 0$.

   (c) Find a sequence $a_k \to \infty$ so that $f(a_k) \to \frac{1}{2}\sqrt{2}$.

   (d) Find a sequence $a_k \to \infty$ so that $f(a_k) \to \frac{1}{2}\sqrt{3}$.

   (e) Can you give an argument that shows that for each y in $[-1, 1]$ there is a sequence $a_k \to \infty$ so that $f(a_k) \to y$?

   (f) Does $f(x) = \sin(x)$ converge as $x \to \infty$? Explain.

3. Does $f(x) = \begin{cases} 1; & \text{for } 2k \le x < 2k+1, \\ 0; & \text{for } 2k-1 \le x < 2k, \end{cases}$ where k represents an integer, converge for $x \to \infty$? Explain.

4. Does $f(x) = \begin{cases} x; & \text{for } 2k \le x < 2k+1, \\ \frac{1}{x}; & \text{for } 2k-1 \le x < 2k, \end{cases}$ where k represents an integer, converge for $x \to \infty$? Explain.

5. Based on the above observations, give a characterization what it means to *not* converge as $x \to \infty$.

Introduction to Limits at a Point

Each of the following functions has a domain which equals the whole real line with one point missing. Find, if possible, a natural, continuous way to extend the definition to the whole real line. Use computational tools for visualization and algebraic computations and verbal arguments for verification and justification.

function	missing point a	proposed value L at a	verification/justification
$f(x) = \dfrac{x^2 + 3x - 28}{x - 4}$			
$f(x) = \sin\left(\dfrac{1}{x}\right)$			
$f(x) = \dfrac{\lvert x - 2\rvert}{x - 2}$			
$f(x) = x \sin\left(\dfrac{1}{x}\right)$			
$f(x) = \begin{cases} x + 1; & \text{for } x < 0, \\ \cos(x); & \text{for } x > 0. \end{cases}$			
$f(x) = \dfrac{1}{x^2}$			

For the functions whose domain you extended, describe the properties of the number L that "fills the gap."

Does it matter for the extension what the behavior of the function is away from a, say outside a little interval $(a - \delta, a + \delta)$, where δ is considered to be a small positive number?

If for the function $f(x) = \dfrac{\lvert x - 2\rvert}{x - 2}$ we restrict our attention to the interval $(-\infty, 2]$, what value L would you propose at a? What value at a would you propose if we restrict attention to the interval $[2, \infty)$?

Practice with limits

Find the limit or show the limit does not exist

1. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

2. $\lim_{x\to -3} \dfrac{x^2+5x+6}{x^2-9}$

3. $\lim_{h\to 0} \dfrac{2(h-1)^2-2}{h}$

4. $\lim_{x\to 4} \left(\dfrac{1}{x-4} - \dfrac{8}{x^2-16}\right)$

5. $\lim_{x\to 2} \dfrac{x-\sqrt{5x-6}}{x^2-4}$

6. $\lim_{x\to \frac{4}{3}} \dfrac{3x-4}{|3x-4|}$

7. $\lim_{x\to 0} f(x)$ where
   $f(x) = \begin{cases} x+1; & \text{for } x > 0; \\ x^2; & \text{for } x \le 0. \end{cases}$

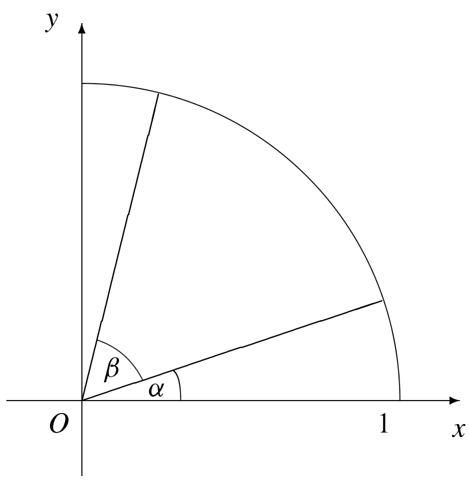
8. $\lim_{x\to 1} f(x)$ for
   $f(x) = \begin{cases} x+1; & \text{for } x \text{ irrational}; \\ 2; & \text{for } x \text{ rational}. \end{cases}$

9. The volume of for a right circular cylinder that is to be designed is given as 22in^3. What happens to the surface-to-volume ratio when r gets close to 0? Answer intuitively and then justify with a computation.

10. If $0 \le f(x) \le x^4$, what can be said about $\lim_{x\to 0} f(x)$? In case $\lim_{x\to 1} f(x)$ exists, what can we say about it?

Deriving the Additive Identity for the Sine Function

In the proof of Theorem 8.2.1 we first prove the additive identity for the cosine function geometrically and then we obtain the additive identity for the sine function via the cofunction identities. The additive identity for the sine function can also be obtained in a direct geometric fashion. This is what we shall do in this activity.



For all sketches etc., please refer to the above figure and add lines, points, notation as necessary. We shall assume that α and β are both angles in the first quadrant.

1. Sketch the angle α in standard position,
2. Sketch the angle β such that its vertex is the origin and such that its initial side is the terminal side of α; note that the terminal side of β also is the terminal side of $\alpha + \beta$ in standard position,
3. Call the point where the terminal side of β intersects the unit circle Q; the y-coordinate of this point is $\sin(\alpha + \beta)$,
4. Sketch a line through Q that is perpendicular to the terminal side of α; call the intersection point of this line with the terminal side of α P,
5. Sketch a vertical line through P and a horizontal line through Q; call the point where these lines intersect R; call the point where the vertical line intersects the x-axis N,
6. Show that the length NP plus the length PR is $\sin(\alpha+\beta)$,
7. Show that the length of OP is $\cos(\beta)$,
8. Show that the length of NP is $\sin(\alpha)\cos(\beta)$,
9. Show that the length of PQ is $\sin(\beta)$,
10. Show that the angle QPR is α,
11. Show that the length of PR is $\sin(\beta)\cos(\alpha)$,
12. Conclude that for all angles α, β in the first quadrant we have $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$.

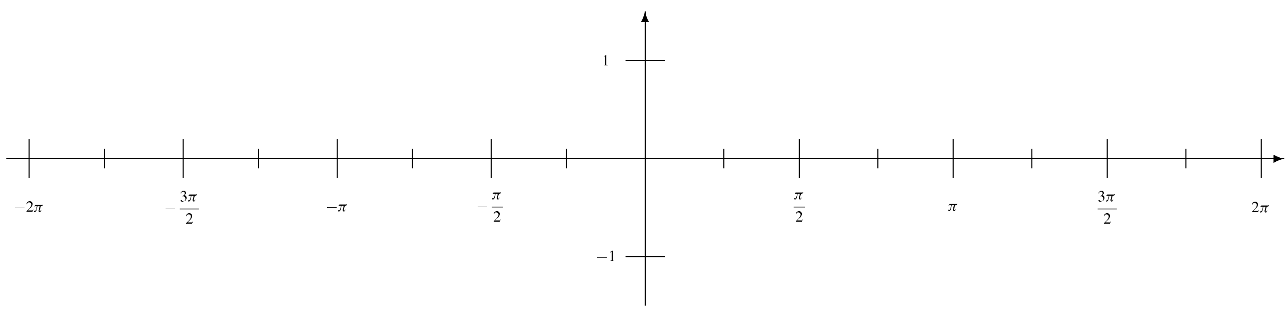
The reminder of the proof is now a technical necessity to show that the identity is indeed valid for all angles in the first four quadrants.

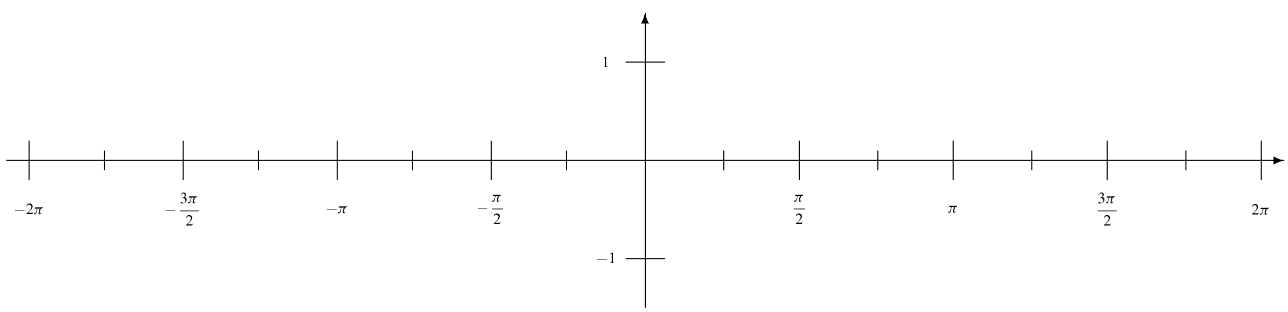
1. Repeat the above argument in the appropriate fashion for α in the first and β in the second quadrant,
2. Use cofunction identities to show $\cos(\beta - \pi) = -\cos(\beta)$ and $\sin(\beta - \pi) = -\sin(\beta)$,
3. Show that the identity holds for α in the first quadrant and β in the third or fourth quadrant by using that if β is in the third or fourth quadrant, then $\beta - \pi$ is in the first or second quadrant,
4. Use similar tricks to show the identity holds for all possible α and β.

Discovering Double Angle Identities

Representations of $\sin^2$ and $\cos^2$.

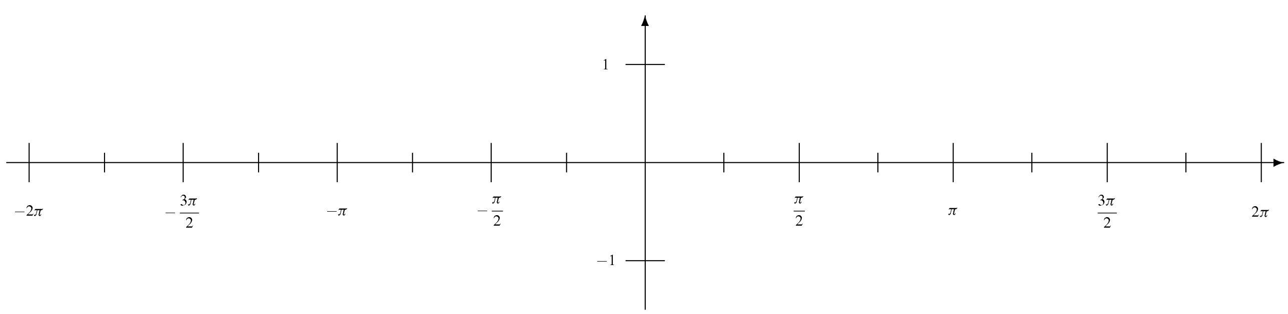
1. Sketch the graph of $f(x) = \sin^2(x)$ on the interval $[-2\pi, 2\pi]$ (CAS allowed).
2. Note that the graph looks like a sped up, shifted harmonic oscillation.
3. Find a, b, d such that $\sin^2(x) = a\cos(bx) + d$.
4. Find a similar representation for $g(x) = \cos^2(x)$.
5. Solve each representation for $\cos(bx)$ to obtain an equation for $\cos(bx)$.





Representing $\sin(x)\cos(x)$.

1. Adapt the above process to obtain a representation for $h(x) = \sin(x)\cos(x)$.
2. Solve the representation for the trigonometric function that is used.



Derivatives Everywhere

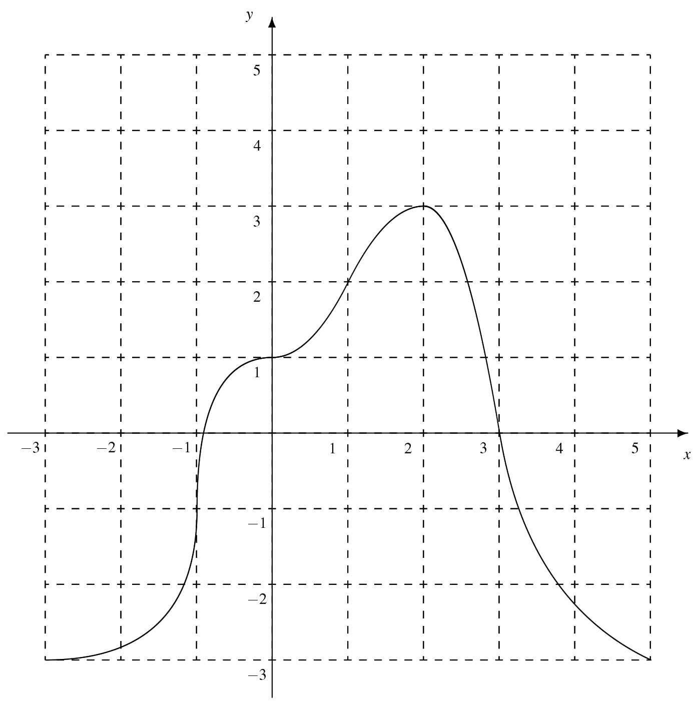
In this activity we answer a variety of questions related to derivatives. The intent is to firm up the knowledge that the derivative as a rate of change can answer a large variety of questions.

1. A parachutist drops out of an airplane at height $5,000\,ft$. The height above the ground as a function of time is given as $h(t) = -16t^2 + 5,000$. Find

   (a) The average velocity over the interval [0, 1],

   (b) The instantaneous velocity at time t.

2. For the function given in the figure below estimate



   (a) The slope of the tangent line at $x = 4$,

   (b) The slope of the secant that intersects the graph at $x = 1$ and $x = 3$,

   (c) $f(-3)$,

   (d) The derivative at $x = -1$.

3. For the function $f(x) = \sqrt{x+3}$ find

   (a) The slope of the secant line through $(1, f(1))$ and $(6, f(6))$,

   (b) The equation of the tangent line through (6, 3).

4. The charge of a laptop battery (in mAh) is charted over the course of a four hour charging period. The values are given in the table below.

time $[h]$	charge $[mAh]$
0	0
0.5	4,000
1	6,000
1.5	7,000
2	7,500
2.5	7,750
3	7,890
3.5	7,950
4	7,980

   (a) Estimate the derivative of the charging function at $t = 1$ and at $t = 3$.

   (b) Is the battery charging faster after one hour or after three hours?

5. Find the meaning of the derivative if f is

   (a) The count of white blood cells over time and $f' > 0$,

   (b) The count of red blood cells over time and $f' < 0$,

   (c) The Dow Jones index over time and $f' > 0$,

   (d) The concentration of salt in a river as a function of the distance from the ocean. Is this derivative positive or negative?

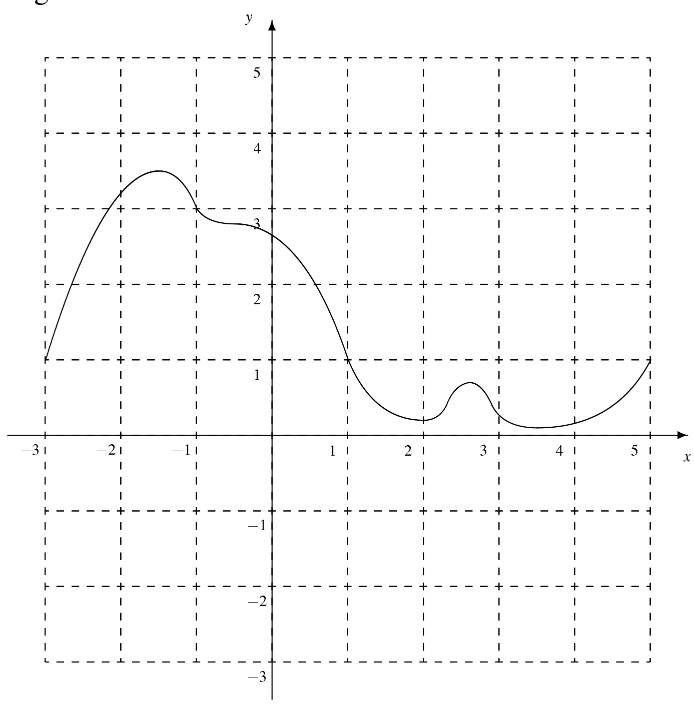
   (e) The density of air as a function of the height above the ocean. Is this derivative positive or negative?

   (f) The average fuel consumption per mile of a car as a function of the speed driven. Is this derivative positive or negative?

6. For the function $f(x) = \dfrac{3x+1}{2-x}$ find the derivative at an arbitrary x.

The Slope at Extrema and Rolle's Theorem

1. For the function in the figure below, sketch the tangent lines at the local extrema.

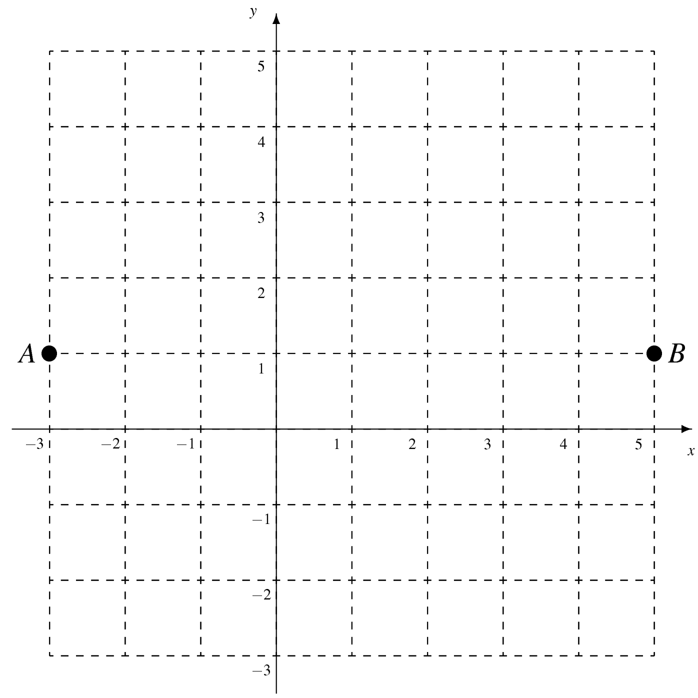


2. What is the derivative at each local extremum?

3. Are these all the places where the derivative has this value?

4. Do you think the derivative at a local extremum must have a certain value? Do you think that this value of the derivative guarantees the existence of a relative extremum?

5. In the figure below attempt to sketch a differentiable function that goes through the two marked points A and B and which does not have any places where the derivative is zero.

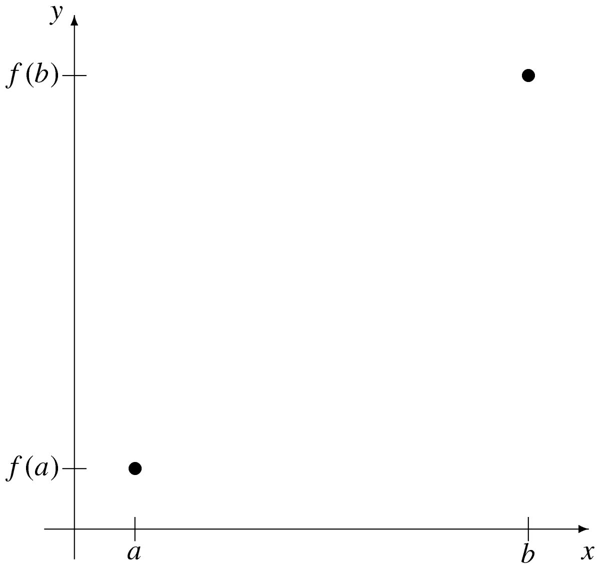


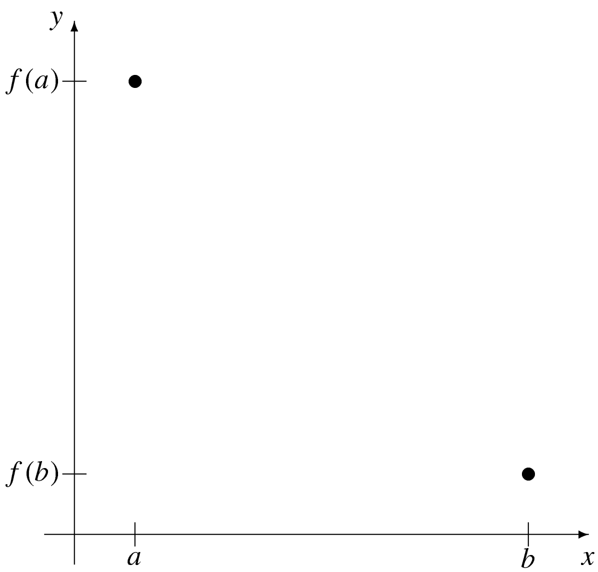
6. Is it possible to sketch such a function? If you think you have sketched such a function, check if the function really is differentiable.

The Mean Value Theorem

Why should the slope of the secant line through $(a, f(a))$ and $(b, f(b))$ be equal to the slope of any tangent line on the interval $[a, b]$? To understand why the theorem works it is instructive to start by attempting to draw a few situations in which the theorem fails.

1. Attempt to sketch two graphs of functions such that you believe that the slope of the secant through $(a, f(a))$ and $(b, f(b))$ (which you should also draw) is *not* equal to the slope of *any* tangent line on the interval $[a, b]$.





2. Now check your drawings. Set the edge of a ruler or of a driver's license parallel to the secant line and move the edge across your sketches. If the edge becomes the tangent line to your graph at some point, then the graph is not as specified in 1 after all. If this is not the case, check if the function you drew is differentiable. (It should not be.) Record your findings for each graph.

3. Conclude that while the above is not a proof, at least it seems very hard to violate the Mean Value Theorem.

4. Now for each graph, tilt the page until the secant line is horizontal. Sketch a new coordinate system such that the secant line is parallel to the x-axis. Apply Rolle's Theorem to find a place where we have a horizontal tangent line in the new coordinate system. Sketch such a horizontal tangent line.

5. Tilting the paper back to the usual orientation we see that we have found tangent lines are guaranteed by the Mean Value Theorem. This shows that rotating the coordinate axes is one way to approach the Mean Value Theorem. If this approach did not work for you, determine what feature of the graph prevented the approach from working.

6. It is formally quite difficult to perform a rotation of the coordinate axes that allows us to apply Rolle's Theorem. (In fact, if our function intersects any line that is perpendicular to the secant line more than once, then it is impossible.) A safer approach is to simply move the second point $(b, f(b))$ to the same height as the first point $(a, f(a))$.

   (a) To do this we need a function that is equal to 0 at a and equal to $f(b) - f(a)$ at b. Find the equation of a straight line with these properties.

   (b) Subtract your straight line from f. Verify that the resulting function g is such that $g(a) = g(b) = f(a)$.

   (c) Apply Rolle's Theorem to g and derive the conclusion of the Mean Value Theorem.

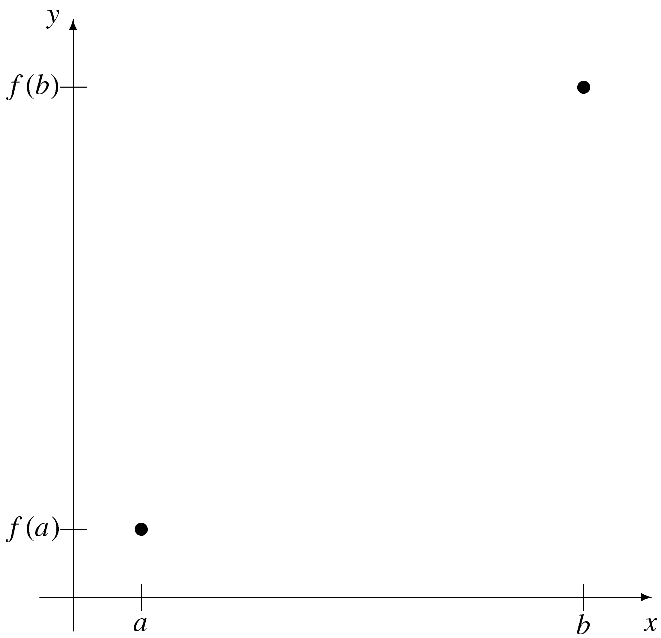
Concavity and the Shape of Graphs

Exploring Concavity

1. In the graph below, draw two differentiable functions that are increasing on $[a, b]$ and which go through the prescribed points $(a, f(a))$ and $(b, f(b))$.

   (a) The graph of the first function should be such that the increase to $f(b)$ happens as early as possible.

   (b) The graph of the first function should be such that the increase to $f(b)$ is delayed as much as possible.



2. What is different about the two graphs? How fast does the slope increase in one graph versus the other?

Using Concavity for Graphing

3. Sketch the graph of the function from the given information.

   (a) $\lim_{x\to-\infty} f(x) = \infty$, $f(-4) = 0$, $f(-2) = 1$, $f(0) = 2$, $f(2) = 1$, $f(4) = 0$, $\lim_{x\to\infty} f(x) = \infty$,

f''	+		+		−		−		+		+
f'	−		+		+		−		−		+
x		−4		−2		0		2		4	

   (b) $\lim_{x\to-\infty} f(x) = 3$, $f(-1) = 1$, $f(0) = 0$, $f(1) = -2$, $f(2) = -3$, $f(4) = -1$, $\lim_{x\to\infty} f(x) = 0$,

f''	−		+		−		+		+		−
f'	−		−		−		−		+		+
x		−1		0		1		2		4	

   (c) $\lim_{x\to-\infty} f(x) = -\infty$, $f(-3) = 0$ (horizontal tangent at $x = -3$), $f(-2) = 2$, $f(0) = 4$, $f(1) = 3$, $f(2) = 2$, $\lim_{x\to\infty} f(x) = \infty$,

f''	−		+		−		−		+		+
f'	+		+		+		−		−		+
x		−3		−2		0		1		2	

4. *If possible*, sketch the graph of the function from the given information. If it is not possible, explain what is wrong.

   (a) $\lim_{x\to-\infty} f(x) = -\infty$, $f(-1) = 2$, $f(1) = 0$, $f(3) = -1$, $f(5) = -3$, $f(7) = -4$, $\lim_{x\to\infty} f(x) = \infty$,

f''	−		−		+		+		+		+
f'	+		−		−		+		−		+
x		−1		1		3		5		7	

   (b) $\lim_{x\to-\infty} f(x) = 0$, $f(-2) = -1$, $f(-1) = -2$, $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $\lim_{x\to\infty} f(x) = 0$,

f''	−		+		+		−		−		−
f'	−		−		+		+		−		+
x		−2		−1		0		1		2	

The second derivative test.

5. Now let us consider places x_0 where the tangent line is horizontal.

   (a) If the function is concave down at x_0, what does the graph look like near x_0?

   (b) If the function is concave up at x_0, what does the graph look like near x_0?

   (c) If the function changes from concave down to concave up at x_0, what does the graph look like near x_0?

   (d) If the function changes from concave up to concave down at x_0, what does the graph look like near x_0?

The Derivatives of the Trigonometric Functions

1. Use a CAS to graphically find the limits $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$ and $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta}$. (Plot both functions near $x = 0$ and determine the values that the functions approach for $x \to 0$.)

2. Use the additive identity $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$ and the limits in 1 to find the derivative of $f(x) = \sin(x)$.
   $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} =$

3. Compare a plot of the function obtained in 2 as the derivative of $\sin(x)$ with a plot of the function
   $$\frac{\sin(x + 0.0001) - \sin(x)}{0.0001}.$$
   Do the two plots look similar? Why should or shouldn't they?

4. Use the cofunction identity $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$ and the chain rule to find the derivative of the cosine function.

5. Represent the tangent function in terms of sine and cosine and find the derivative of the tangent function.

6. If so desired, find the derivatives of the secant, cosecant and cotangent functions.

Practice for applying the differentiation rules.

7. Find the derivative of
   (a) $f(x) = \sin^2(3x + 2)$
   (b) $f(x) = x\sin(2x)$
   (c) $f(x) = x\cos(x)e^x$

8. Find where the function is increasing or decreasing.
   (a) $f(x) = \cos^2(x)$
   (b) $f(x) = e^{-x}\sin(3x)$

9. Find the 34^{th} derivative of $\cos(x)$.

10. Find the n^{th} derivative of $\sin(x)$.

Practice with Product Rule, Quotient Rule and Derivatives of Trigonometric Functions

Find the derivative

1. $F(x) = \frac{5}{x} - 3\sqrt{7x}$
2. $F(x) = \frac{2x^{\frac{9}{4}} + 7x^{-3}}{-5x^2 + 3x^{\frac{1}{2}}}$
3. $F(x) = 3x^5 \tan(x) + x$
4. $F(x) = \sin(x)\cos(x) + \tan(x) + 1$
5. $F(x) = x^2 \cos(x)\cot(x) - 5$
6. $F(x) = \frac{5x\left(3\sqrt{x} - 2\sin(x)\right)}{\cos(x)\sin(x)} + 2x$

Find the limit

1. $\lim_{x\to 0} \frac{\sin(5x)}{3x}$
2. $\lim_{x\to 0} \frac{\cos(2x)\tan(2x)}{7x}$
3. $\lim_{x\to 4} \frac{\sin(x-4)}{x^2 - 2x - 8}$

The van der Waals equation of state for an ideal gas is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT,$$

where P is the pressure, V is the volume that a mole of the gas would occupy, $R \approx 8.31441 \frac{J}{mol \cdot K}$ is the universal gas constant, T is the temperature of the gas. Find $\frac{dP}{dV}$ when T is constant. Interpret $\frac{dP}{dV}$.

Practice with the Chain Rule

Find the derivative

1. $F(x) = \sin(5x) + \sqrt{x^2+1}$
2. $F(x) = \sqrt{\tan(x)}(x^2+2) - 3$
3. $F(t) = \sqrt{\dfrac{5t^4+1}{2t^2+3t+9}} + 7x$
4. $F(x) = \sin(\cos(x)) + \cos(\sin(5x+2))$
5. $F(x) = \sin(\cos(\tan(3x)))$
6. $F(x) = \cos^3(\sin^2(x))$
7. $F(x) = [h(\cos(x))]^2$, where h is a differentiable function

1. Find the equation of the tangent line of

$$f(x) = 10\sin^2(5x)$$

at $a = 0$ and at every multiple of π.

2. The radius of a given sphere increases at a rate of

$$\frac{dr}{dt} = 2.5\frac{\text{cm}}{\text{s}}.$$

At what rate is the surface area increasing when $r =$ 20cm?

3. A spherical balloon is filled at a constant rate of

$$\frac{dV}{dt} = 10\frac{\text{in}^3}{\text{min}}.$$

At what rate is the radius of the balloon increasing when $r = 2in$?

Discovering l'Hôpital's Rule.

In the first part of this activity we shall investigate two situations in which the evaluation of a limit leads to the indeterminate form "$\frac{0}{0}$."

1. Find the equation $y_1(x)$ of a straight line that is zero at $x = a$ and has slope m.

2. Find the equation $y_2(x)$ of a straight line that is zero at $x = a$ and has slope n.

3. Find $\lim_{x \to a} \dfrac{y_1(x)}{y_2(x)}$.

Now consider the function $f(x) = \dfrac{\sin(3x)}{xe^{2x}}$.

1. Graph $g(x) = \sin(3x)$ and $h(x) = xe^{2x}$ near the origin and zoom in. What do the graphs of both functions look like under sufficiently high magnification?

2. So what should the limit $\lim_{x \to 0} f(x)$ be?

3. Check your conjecture graphically by graphing $f(x)$ near the origin.

Working with l'Hôpital's Rule.

Compute the limits:

1. $\lim_{x \to 0} \dfrac{\sin(3x)}{7x}$

2. $\lim_{x \to 1} \dfrac{\ln(x)}{\sqrt{x-1}}$

3. $\lim_{x \to \infty} \dfrac{x^4}{e^{2x}}$

4. $\lim_{x \to \infty} x \sin\left(\dfrac{1}{x}\right)$

5. $\lim_{x \to 0} \left(\dfrac{1+3x}{\sin(x)} - \dfrac{1}{x}\right)$

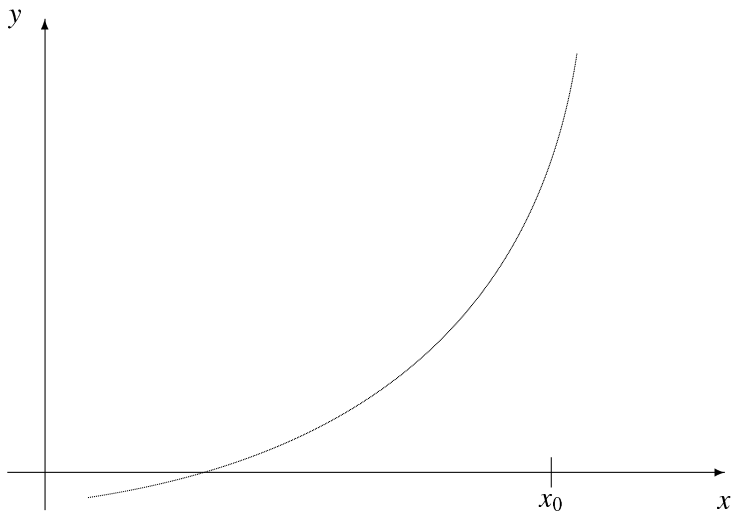
6. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

7. $\lim_{x \uparrow \frac{\pi}{2}} (\cos(x))^{4\cos(x)}$

8. $\lim_{x \downarrow 0} \left(\dfrac{1}{x}\right)^{x^2}$

9. $\lim_{x \to \infty} \dfrac{\sqrt{x^2-1}}{x}$

Finally find the domain of the function in 7.



Discovering Newton's Method

For the function given in the figure perform the following procedure repeatedly:

1. Draw the tangent line to the graph at x_0.
2. Find the x-intercept of the tangent line, call it x_1.
3. Repeat the above steps with x_1 in place of x_0 to obtain x_2, then with x_2 to obtain x_3, etc.

What value do the values x_1, x_2, x_3, etc. approach?

Reflecting the above graphical discovery symbolically, let f be a differentiable function and let x_n be a point on the x-axis.

1. Find the equation of the tangent line to f at x_n.
2. Find the x-intercept of this tangent line and call it x_{n+1}.

Working With Newton's Method

1. Find 8-digit approximations to the zero(es) of

   $$f(x) = xe^x - 1$$

   Record your starting point and how many iterations were needed. Would a different choice of the starting point speed up the process?

2. Find 8-digit approximations for the solutions of $\cos(x) = x$.

3. Explain what happens when you execute Newton's method for $f(x) = x^3 - 3x + 3$ with $x_0 = \frac{3}{2}$. Is there a way to fix this problem?

Discovering Substitution

Consider the indefinite integral

$$\int e^{-\frac{x^2}{2}} x \, dx$$

1. Identify what makes the integral hard to solve.

2. Let $u := -\frac{x^2}{2}$ and compute $\frac{\mathrm{d}u}{\mathrm{d}x}$.

3. Replace all quantities in the indefinite integral that involve an x with quantities that involve u and no x anymore.

4. Solve the integral.

5. Replace all occurrences of u with the appropriate expression involving x.

6. Verify that the function thus obtained is an antiderivative of $e^{-\frac{x^2}{2}} x$.

7. Why does this work?

Working with Substitution

Find the integral. For definite integrals use the

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

index shifting method rather than back substitution.

1. $\int \cos(4x) \, dx$
2. $\int \left(3x^2 + 1\right) e^{3x^3+3x} \, dx$
3. $\int_1^2 \sqrt{x-1} \, dx$
4. $\int_{\ln 3}^{\ln 10} \cos\left(e^x + 10\right) e^x \, dx$
5. $\int \frac{\cos(x)}{1+\sin^2(x)} \, dx$
6. $\int \frac{1}{x \ln(x)} \, dx$

   (Trickier ones follow.)

7. $\int \tan(x) \, dx$
8. $\int_1^{64} \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}} + 2} \, dx$
9. $\int \frac{x^5}{\sqrt[5]{x^2+4}} \, dx$
10. $\int_{-1}^{1} \left(1 - x^2\right)^{\frac{3}{2}} \, dx$ (hint: use a substitution $x = h(u)$)

Practice with integration by parts.

Find the integral

1. $\int t\cos(t)dt =$

2. $\int x^2 e^x dx =$

3. $\int \frac{\ln|x|}{x^2} dx =$

4. $\int e^{2x}\cos(x)dx =$

5. $\int \ln|x| dx =$

6. $\int x^2 (\ln|x|)^2\, dx =$

7. $\int x\tan^{-1}(x)dx =$

8. $\int x^2\tan^{-1}(x)dx =$

9. $\int x\left(2+x^2\right)\ln\left(2+x^2\right) dx =$

10. $\int \cos\left(\sqrt{x}\right) dx =$

11. $\int x\ln|3x| dx =$

Integration of Rational Functions Through Partial Fraction Decomposition.

Discovering Partial Fractions.

Integrating polynomials is a simple application of the power rule. The next larger class of functions that contains polynomials is the class of rational functions. *In principle*, it is possible to find the antiderivative of any rational function. The key is to decompose the function into pieces that we can handle. Let us first take stock of rational functions that we can integrate.

1. $\displaystyle\int \frac{1}{x-c}\,dx =$
2. $(n > 1,)\displaystyle\int \frac{1}{(x-c)^n}\,dx =$
3. $\displaystyle\int \frac{x}{x^2+b^2}\,dx =$
4. $(n > 1,)\displaystyle\int \frac{x}{\left(x^2+b^2\right)^n}\,dx =$
5. $\displaystyle\int \frac{1}{x^2+b^2}\,dx =$
6. $(n > 1,\ \text{hard})\displaystyle\int \frac{1}{\left(x^2+b^2\right)^n}\,dx =$

Integration by partial fraction decomposition consists of the decomposition of the function in question into pieces as above and subsequent integration of the pieces. Steps are as follows.

1. Perform a long division with the rational function to obtain a polynomial part and a remainder part where the degree of the numerator is smaller than the degree of the denominator. The polynomial part is easily integrated, which leaves the remainder.

2. Factor the denominator of the remainder part into linear factors and irreducible quadratic factors, that is, quadratic factors without any real zeros. (In practice it is not always possible to find this factorization. High degree polynomials can be problematic here.)

3. For each linear factor $(ax+b)^n$ in the denominator generate a term $\dfrac{A_k}{(ax+b)^k}$, where k ranges from 1 to n.

4. For each irreducible quadratic factor $\left((x-a)^2+b^2\right)^n$ in the denominator generate a term $\dfrac{B_k x + C_k}{\left((x-a)^2+b^2\right)^k}$, where k ranges from 1 to n.

5. Set the sum of all these terms equal to the remainder part.

6. Obtain a system of equations for the unknown constants A_k, B_k, C_k by setting corresponding coefficients equal. (Can be combined with Heaviside's method.)

7. Solve the system of equations and plug the result into the sum in part 5. This is the partial fraction decomposition of the remainder part. The parts of the decomposition can now be integrated directly or by using the above formulas or by using an integral table.

Working with partial fractions.

1. $\displaystyle\int \frac{x-3}{(x-1)(x+3)}\,dx =$
2. $\displaystyle\int \frac{x^3-2x}{x^2+3x+2}\,dx =$
3. $\displaystyle\int \frac{x-3}{x^3-x^2-x+1}\,dx =$
4. $\displaystyle\int \frac{2x^2-x+4}{x^3+4x} =$
5. $\displaystyle\int \frac{x^2}{\left(x^2+4\right)^2}\,dx =$

Mixed Practice with Symbolic Integrals

Solve the integral

1. $\int 5x\cos(x)dx$
2. $\int_0^1 xe^{x^2}dx$
3. $\int_{-1}^1 \frac{3x\cos(x)}{12-x^4}dx$
4. $\int xe^{3x}dx$
5. $\int 5x\cos\left(x^2\right)dx$
6. $\int \frac{2}{4+x^2}dx$
7. $\int \frac{4x}{1+x^2}dx$
8. $\frac{1}{(x-1)(x+1)}dx$
9. $\int \cos(x)\sin^2(x)dx$
10. $\frac{1}{(x-1)\left(x^2+1\right)}dx$
11. $\int \cos(x)\sin(x)dx$

Find the integral using an integral table.

1. $\int \sqrt{x^2-2x+10}dx$
2. $\int \sin^3(x)\cos^4(x)dx$
3. $\int \frac{\sqrt{3-7x}}{x^2}dx$
4. $\int x^5\ln(x)dx$
5. $\int e^{3x+1}\cos(11x)dx$

Computing the Volume of a Tetrahedron

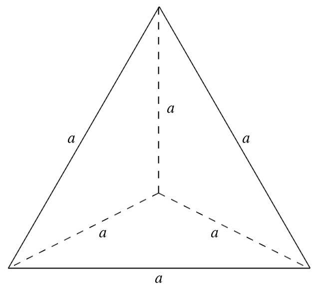


Figure ACT.3: View of a tetrahedron looking straight at one of the faces (dashed lines are hidden lines).

The tetrahedron (cf. Figure ACT.3) is one of the five Platonian solids. (The Platonian solids are solid figures such that every face is a regular polygon.) All faces of the tetrahedron are equilateral triangles as shown in the sketch above. Computing its volume is a good example of a sophisticated analysis that requires us to consider several different views of the object.

1. Determine the shape of the horizontal cross sections of a tetrahedron.
2. Derive or otherwise reproduce the formula for the area of the cross section as a function of the side length. Use Figure ACT.4.

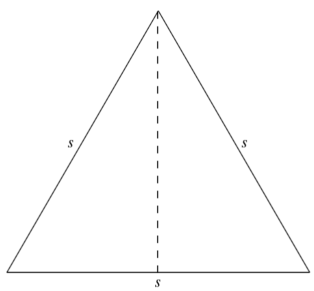


Figure ACT.4: An equilateral triangle.

3. Switch views to a view of one of the sides of the tetrahedron such that we see the bottom edge-on (cf. Figure ACT.5).

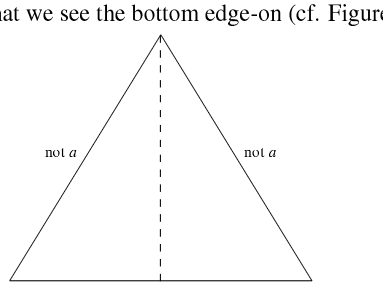


Figure ACT.5: Viewing a tetrahedron such that we see the bottom edge-on.

   (a) Explain how we need to move the tetrahedron in Figure ACT.3 to obtain the positioning in Figure ACT.5.
   (b) Explain why in Figure ACT.5 the sides that go to the top are not sketched at length a.
   (c) Explain why we can find the equations of the lines that bound the side if we can find the height of the tetrahedron.
   (d) Explain why we can find the side length of the cross sections if we know the equations of these lines.

4. Switch views to view the tetrahedron from the plane on which the bottom rests in such a way that the bottom and one of the sides are viewed edge-on (cf. Figure ACT.6).

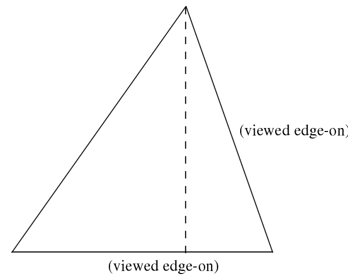


Figure ACT.6: Viewing a tetrahedron so that we see the bottom and the right face edge-on.

   (a) Explain how we need to move the tetrahedron in Figure ACT.5 to obtain the view in Figure ACT.6.
   (b) Explain why the hypothenuse of the smaller triangle on the right is as long as an equilateral triangle with side length a is high.
   (c) Explain why the horizontal side in the smaller triangle on the right is as long as the distance of the center point of an equilateral triangle from one of the sides.

5. Determine the distance of the center point of an equilateral triangle of side length a from one of the sides. To do so, consider Figure ACT.7.

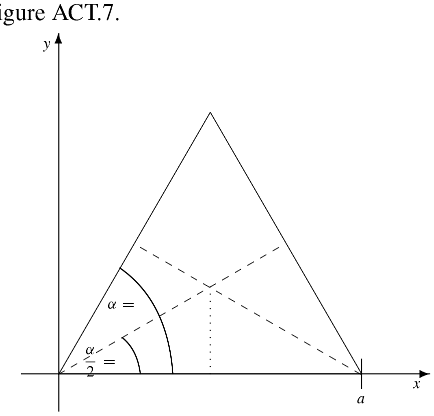


Figure ACT.7: Computing the center of an equilateral triangle.

   (a) In the equilateral triangle above, determine the angles α and $\frac{\alpha}{2}$.
   (b) Use $\frac{\alpha}{2}$ to determine the slope of the dotted line that goes through the origin.
   (c) Use symmetry to find the x-coordinate of the center point.
   (d) Find the y-coordinate of the center point, which is the distance of the center point from any of the sides.

6. Compute the height of the tetrahedron.
7. Compute the side length of each cross section.
8. Compute the volume.
9. Compute the volume of a tetrahedron that is truncated at half its height.

Improper Integrals

Discovering Improper Integrals

There are two types of improper integrals, here introduced by example.

1. *Integrals over infinite intervals.* When trying to compute the area under $f(x) = \dfrac{1}{x^2}$ from $x = 1$ to ∞ we realize that the base has infinite length. But does this mean the area must be infinite?
   (a) Name a situation in which this integral may arise.
   (b) Geometrically we can see that the integral is less than the sum
   $$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$
   and more than
   $$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots,$$
   but that only helps if we know how to compute infinite sums.
   (c) What can we compute?
   $F(t) := \displaystyle\int_1^t \frac{1}{x^2}\, dx =$
   (d) What should the upper bound of the integral be for the area we want to compute?
   (e) If we want to "evaluate" F at ∞, what tool do we normally use?
   (f) Use this tool to find the area.
   (g) **A Standard Improper Integral** (The value of this improper integral depends on the value of p.)
   $$\int_1^\infty \frac{1}{x^p}\, dx =$$
2. *Integrals up to or across singularities.* Consider the area under $f(x) = \dfrac{1}{\sqrt{x-3}}$ from 3 to 5.
   (a) What is "wrong" with this area?
   (b) How can we compute it or show it is infinite?
   (c) **A Standard Improper Integral (the "other end")** (The value of this improper integral depends on the value of p.)
   $$\int_0^1 \frac{1}{x^p}\, dx =$$

Computing Improper Integrals

1. $\displaystyle\int_0^\infty e^{-5x}\, dx =$
2. $\displaystyle\int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx =$
3. $\displaystyle\int_0^\infty \frac{1}{1+x^2}\, dx$
4. $\displaystyle\int_0^\infty xe^{-x}\, dx =$
5. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\, dx =$

Estimating Improper Integrals

Estimate the value of the given integral to the given accuracy ε if the integral converges. If the integral diverges, so state.

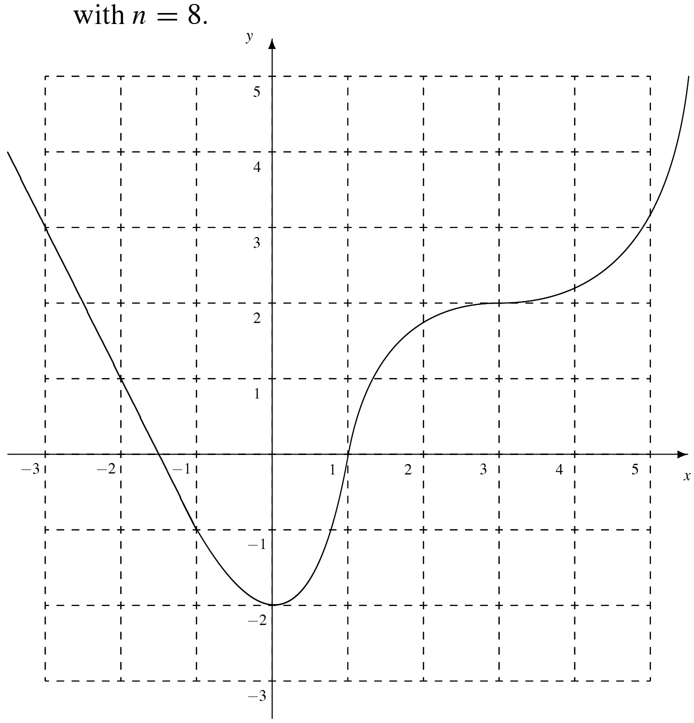
1. $\displaystyle\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx,\ \varepsilon = 0.0001$
2. $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\, dx,\ \varepsilon = 0.001$
3. $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2}\, dx,\ \varepsilon = 0.005$
4. $\displaystyle\int_1^\infty \frac{\ln(x)}{1+x^3}\, dx,\ \varepsilon = 10^{-2}$
5. $\displaystyle\int_1^\infty \frac{e^x}{x^{10}}\, dx,\ \varepsilon = 0.005$
6. $\displaystyle\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx,\ \varepsilon = 10^{-4}$

Numerical Integration

Using the Integration Formulas

Compute approximations for the given integrals using the midpoint rule, Simpson's rule, and the trapezoidal rule with the given number n of partitions.

1. $\int_0^2 e^x \, dx, n = 6$
2. $\int_1^5 \sin(x) \, dx, n = 40$
3. $\int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx, n = 10$
4. $\int_1^2 \cos\left(x^2\right) dx, n = 20$
5. The integral from 1 to 4 for the following function with $n = 8$.



For those integrals that can be solved exactly, compare the approximations with the exact answer. Based on your computations, which technique appears to be the most accurate? That is,

- Which technique gives the smallest error bound when the number of partitions is kept equal?
- Or equivalently, for a given targeted error, for which technique do we need the smallest number of partitions?

Error Estimates

For each of the following integrals

- If the number n of partitions is given, find an upper bound on the error made when approximating the integral using the techniques exhibited in the text (most likely trapezoidal, midpoint and Simpson),
- If an upper bound for the error $|E|$ is given, find the smallest number n of partitions that is guaranteed to keep the error below the given bound when approximating the integral using the techniques exhibited in the text.

That is, you will have as many answers for each problem as we have techniques (with error formulas) available.

1. $\int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx, n = 10$
2. $\int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx, |E| \leq 0.0001,$
3. $\int_1^2 \cos\left(x^2\right) dx, |E| \leq .0001,$

Based on your computations, which technique appears to be the most accurate? That is,

- Which technique gives the smallest error bound when the number of partitions is kept equal?
- Or equivalently, for a given targeted error, for which technique do we need the smallest number of partitions?

Taylor Polynomials

Polynomials that approach e^x.
`(CAS recommended.)`

1. Plot the function $f(x) = e^x$ and several sums of the form $\sum_{n=0}^{k} \frac{1}{n!} x^n$ in the same window.

2. Describe how the difference between the exponential function and the sum $P_k(x) := \sum_{n=0}^{k} \frac{1}{n!} x^n$ changes as k increases.

3. Compare the first k derivatives at $x = 0$ of $f(x) = e^x$ and

$$P_k(x) := \sum_{n=0}^{k} \frac{1}{n!} x^n.$$

Attempting to use the same effect for other functions.

Pick a function that is differentiable at 0. ($\sin(x)$, $\cos(x)$, e^{x^2} are possible choices.)

1. Compute its first k derivatives.

2. Determine a polynomial of degree k that has the same first k derivatives at 0 as your function.

3. Plot the function and the polynomials for increasing values of k. Describe what happens to the difference between the function and the polynomials.

Error Estimates for Taylor Polynomials

The Function $\frac{1}{1-x}$.
`(CAS recommended.)`

1. Find the Taylor series about zero of the function $f(x) = \dfrac{1}{1-x}$.

2. Plot the function $f(x) = \dfrac{1}{1-x}$ and several partial sums
$$T_N(x) = \sum_{n=0}^{N} c_n x^n$$
of its Taylor series about 0 in the same window for $-2 \le x \le 2$.

3. Do you think it is possible to choose N large enough so that $|f(x) - T_N(x)| \le \dfrac{1}{2}$ for all $-2 \le x \le 2$?

Estimating Errors.

Consider the function $f(x) = \sin(x)$ on the interval $[-2, 2]$ and its Taylor polynomials T_N about 0.

1. Determine an upper bound for the error $|f(x) - T_N(x)|$ on $[-2, 2]$ for $N = 5$.

2. Determine an N such that the error $|f(x) - T_N(x)|$ on $[-2, 2]$ is at most 10^{-10}.

The Shape of the Exponential and the Normal Distribution.
On the interval [0, 5], sketch the graph of an exponential density function for various values of the parameter Θ. Record your sketches below.

Describe the general shape of the curves (this can be done in two words). How does increasing the parameter Θ change the shape of the curve? Write/speak/think in terms of where most of the mass is concentrated.

On the interval $[-5, 5]$, sketch the graph of a normal density function with $\sigma = 1$ for various values of the parameter μ. Record your sketches below.

Briefly describe the general shape of the curves (this can be done in two words). Describe how varying the parameter μ changes the shape of the curve. Write/speak/think in terms of where most of the mass is concentrated.

On the interval $[-5, 5]$, sketch the graph of a normal density function with $\mu = 0$ for various values of the parameter σ. Record your sketches below.

Describe the general shape of the curves (this can be done in two words). How does increasing the parameter σ change the shape of the curve? Write/speak/think in terms of where most of the mass is concentrated.

Mean and Variance of a Normally Distributed Random Variable.
Recall that the density of the family of normal distributions is

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

for $-\infty < x < \infty$ and that

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

and

$$V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x) dx.$$

To determine the expected value of a normally distributed random variable with parameters μ and σ first set up an integral for $E(X)$. First use a substitution to remove the μ from the exponent of the exponential function. Second, use a substitution to remove the σ from the exponent of the exponential function. Then solve the remaining integrals. Use $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

To determine the variance follow the same steps as outlined for the expected value.

Choosing the Mean to Achieve a Certain Probability

The hardness of steel is measured in Rockwell hardness units, which are dimensionless quantities. Suppose RheinStahl's equipment allows the production of steel rods with a given hardness with a standard deviation of 1 Rockwell unit. Also assume the hardness is normally distributed around its mean.

How do we need to choose the mean to make sure that 99% of all rods have a hardness of at least 58?

The waiting time for a customer service representative at FastLittleOnes computer corporation is exponentially distributed with mean Θ. What mean waiting time Θ should the company strive to achieve if they want less than 2% of their customers to wait for more than 3 minutes?

Generating Random Data with a Given Distribution

This activity exhibits how random data can be generated on a computer. Such "artificial data" is useful when testing new statistical procedures for their sensitivity to changes in the distribution of the population.

Your spreadsheet should have a function called `rand()` or similar. Such a function returns a floating point random number between zero and one every time it is called. The generation of such random numbers[5] is an important subject in itself which we will not touch upon here. Generate a column of 200 random numbers using RAND and produce the histogram of the data obtained. The data should look like data sampled from a uniform distribution on the interval $(0, 1)$.

To turn sampled data from a uniform distribution and turn it into sample data from another distribution consider the following. If U is uniformly distributed on $(0, 1)$, then

$$P(U \leq u) = u.$$

Thus for any function f into $[0, 1]$ (and in particular for cumulative distribution functions F_X of random variables X) we have

$$P(U \leq f(x)) = f(x).$$

So if F_X is a cumulative distribution function that has an inverse, then

$$P\left(F_X^{-1}(U) \leq x\right) = P(U \leq F_X(x)) = F_X(x).$$

We have arrived at the following.

Theorem ACT.3.1 (The probability integral transform.) *If F_X is a strictly increasing function into $(0, 1)$ and U is a uniformly distributed random variable on $(0, 1)$, then $F_X^{-1}(U)$ is a random variable with distribution function F_X.*

Hence, if we can compute and invert the cumulative distribution function of a random variable X, we can simulate its behavior on a computer via values $F_X^{-1}(U)$.

To be submitted: Spreadsheet file with 200 random data points generated from the distribution with probability density function $f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$. Printout of a histogram of the data.

[5] The generated numbers are actually pseudo-random numbers, since the algorithm that generates them is deterministic. What gives the impression of randomness is that the pattern generated is so complex as to be considered random. To increase the randomness many such algorithms also start out with a random "seed" number that can be specified by the user. Sometimes the seed number itself can be specified as being, say, the hundredths digit of the internal clock at the time of startup of the program. In this fashion, data indistinguishable from random data can be obtained.
Before computer technology people either used the algorithm to compute the numbers by hand, or they used random number tables. Imagine that; a book filled with random numbers! But in their day such books had their use.

Using the Central Limit Theorem.

A car manufacturer wants to obtain an estimate on a new model's fuel consumption. 20 cars are driven in regular traffic and their fuel consumption is measured. The average mileage per gallon is determined to be $25mpg$. Long term observations from similar models and taking different driving habits into account suggest a standard deviation of $2mpg$. What is the probability that the actual mean mileage per gallon is less than 24mpg?

What is the probability that the actual mean mileage per gallon is between $25mpg$ and $28mpg$?

How many cars should have been tested to obtain a measured average that is with 90% probability within 1mpg of the actual mean?

Comparing z_α and t_α^n

The values z_α and t_α^n are useful in probabilistic computations. The t_α^n can be used in situations in which the Central Limit Theorem does not apply.

This activity investigates the fact that the t-distribution approximates the normal distribution as n gets large. Aside from the theoretical merit of such an activity, knowing how fast the t_α^n approach the z_α gives a rough idea what a small sample size is. One could consider any sample size n for which z_α and t_α^n differ significantly as "small".

1. Make a table of the values of z_α and t_α^n (where n is in an absolutely referenced cell, so that we can adjust it later). Set it up in the interval [0, 1] with step length .01.

2. Compare the values of z_α and t_α^n for the table you generated. This can be done by computing the difference between the two in a third column or by plotting the two data columns as charts.

3. For what value of n do all differences become smaller than .1? Smaller than .01? Smaller than .001?

4. Based on your experimentation give a value for n for which you consider the t-distribution replaceable with the standard normal distribution in computations. Justify your answer.

Estimating the Mean of a Given Population

In this activity we use the (unsorted!) data that was used in the activity on page ACT-4 as the underlying population. We can compute the actual mean and variance, and we will use them for control purposes. However, in this activity we shall pretend that the actual mean and variance are unknown. This is done to simulate a real sequence of measurements. In a real measurement the exact value of a quantity either is impossible to measure in principle (say the exact time it takes a pendulum to perform 10 oscillations), or it is impractical to measure (for example it is impractical for most purposes to survey every resident of the USA on their opinion on a certain issue).

Our primary goal is to get an estimate on the mean of our population that is with 95% probability within .5 of the actual mean. This does *not* mean the value we eventually report *must* be within .5 of the actual mean. There is a 5% chance that it won't be. In an actual measurement we would not be able to detect this incident, as we do not know the actual mean of the population. An added real-life difficulty is that we also do not know the standard deviation of our population. Thus we also need to use estimates for the standard deviation instead of actual values.

Our secondary, but almost equally important, goal is to obtain our estimate with as few measurements as possible. This is a reflection of the fact that measurements come at a cost in terms of employee time and usage of resources. One usually tries to minimize these costs. To simulate a real life measurement we allow ourselves

- To *only* use the first n values in the column.

  In a real life situation one can use the first n measurements in a sequence of possible measurements.

- To increase the number n of measurements if necessary.

  In a real life situation we are allowed to take more measurements if an initial sequence of measurements does not lead to the desired accuracy.

First Approach: We do not know the standard deviation of our population. What we do know is that the population consists of numbers between 0 and 11. Thus we could estimate $\sigma \leq 11$ as an extremely crude estimate. Your first task is to use this estimate to give a lower bound for n that guarantees the desired accuracy of the sample mean with the desired probability.

The estimate for n you obtain is quite large. Thus, while the estimate for n is guaranteed to satisfy our first goal (why?) it does not satisfy the second goal. One way to improve the estimate for n is to find a better estimate for the standard deviation. This can be done theoretically or via the following adaptive procedure.

Second Approach:

1. Take the first 10 values and compute their sample mean and sample standard deviation.

2. Use the sample standard deviation as an estimate for the standard deviation σ of the population. Use it to compute the probability that our sample mean is within .5 of the actual mean. This can and should be implemented as formulas that depend on the sample size n. Most likely this first probability is smaller than 95%.

3. Increase n to 11 and recompute all values. Put the new computations into a new column. In this fashion you will keep track of previous work. Also, if the right set of absolute and relative addresses is used, you will only need to copy your previous computations and change n in the appropriate places.

4. Continue this process until your estimate of the probability that the sample mean is within .5 of the population mean is $> 95\%$ for 5 times in a row.

5. Record your estimate for the mean and compare it to the actual mean.

To be submitted.

- Spreadsheet with record of the work.

- Last value for n that was used to compute the sample mean. Estimated value for μ that is with 95% probability within .5 of the actual mean.

- Answer the following questions.

  1. How do your estimates of the mean and the standard deviation compare to the actual values μ and σ?
  2. Why do you think we wanted to have a "good" estimate ($> 95\%$ chance the mean is within .5 of the mean) five times in a row before reporting an estimate?

Designing Confidence Intervals

A new character recognition program has been tested 20 times on sets of 500 characters. On average the program recognized 450 of the characters correctly with a standard deviation of 25 characters. Compute a 99% confidence interval for the number of correctly recognized characters (out of 500).

How many sets of 500 characters would need to be tested to obtain a 99% confidence interval on the number of recognized characters (out of 500) with length 10?

The Comparison Test

Discovery.

Often it is possible to determine if a series is convergent or divergent by comparing it to another series whose convergence behavior is known.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

1. Prove that for all n we have

$$\frac{1}{\sqrt{n}} \geq \frac{1}{n}.$$

2. Prove that for all k we have

$$\sum_{n=1}^{k} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{k} \frac{1}{n}.$$

3. Given what we know about the harmonic series, is it reasonable to assume that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges?

4. Is it possible to use a similar comparison to determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges?

Practice.

Apply the comparison test (Theorem 19.2.1) to determine if the following series converge or diverge. You may compare the series to any of the series in Section 19.2.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

2. $\sum_{n=2}^{\infty} \frac{1}{\log_2(n)}$

3. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+4n}$

4. $\sum_{n=1}^{\infty} \frac{n+2}{n^3}$