

# Appendix E

## Instructor's Guide

This instructor's guide grew out of my class preparations for teaching differential equations. Over the years, for many sections, I accumulated more examples and suggestions than could or should be covered in class. For other sections, I can't think of much more to say than "present the content". So this guide certainly is food for thought and a source for examples that are not in the book, but it cannot prescribe how to run a class. (Students who find this guide on the web should take this remark to heart: Your teacher is not bound in any way to follow these notes.) To be efficient, and because in a computational class, the approach is quite canonical, I stuck with a small number of headings.

**Suggested Time.** Classes are assumed to last 75 minutes. This is the standard class length I teach, so it would need to be scaled for 50 minute classes. But I have the feeling that some of my presentations can be compressed. (For example, my videos are shorter than my classes.)

**Lecture/Presentation.** Contains some bullets on what could be told in lecture. Examples are usually split out into the group work section.

**Group Work/Examples.** Contains examples that can be presented on the board or worked in groups. (Also see Section E.1 on possible structures for group work.)

Note that I often use activities from Appendix D to get students started on a topic, then I explain what I think needs explaining and then I let students practice some more. (In this fashion, some of the activities actually are my class preparation.) During discovery and practice, I look over students' shoulders, giving suggestions, correcting mistakes etc. (In a small class, I'll try to visit every student. When pressed for time or in larger classes, I visit different segments on different days until the whole class is covered.) These observations and conversations with students tell me how far I should go in explanations before we reach a "sticky point". After I have observed enough, I start putting solutions on the board (but I won't ask anyone to follow me, because everyone will typically have the early part). Once I have enough material on the board, we quickly run through it and once what's on the board is done, it's either back to practice or I continue lecturing. This approach allows students to double check if their work is right, and it allows students who were stuck to still complete the tasks.

**Reading Quiz.** These questions have been used as (parts of) reading quizzes. They can (at least) serve as an inspiration for other reading quizzes. Although the questions are simple, it should be remembered that reading quizzes should only test basics. It should also be said that when sections overlapped, parts of reading quizzes from several sections were combined.

**Notable Homework Problems.** Mentions homework problems that are worth pointing out. What I find notable may not be notable to everyone and some notable problems may well not be listed, but nonetheless.

*As with all materials that I make available on the web, although it would be nice if this guide was used with my text, it can be used in conjunction with any other text, too.*

For some of the project sections, I have rubrics that I use to grade projects. For me, they make the projects a bit easier to grade.

Not all headings will have notes in all sections. Remember that this is not a commercial supplement that tries to be all things to all people. It's what I did/do/would do, and I have found that with enough preparation, I often don't use notes and just teach, referring to this guide for examples that touch upon salient points and for which the numbers don't work out too badly. Similar to the comments about active learning of abstract content at the start of Appendix D, being prepared for class does not mean to have notes on paper, it means having a plan in your brain.

Despite the rather blunt comment at the start of Appendix D, I have seen benefit in a variety of active learning techniques. As long as the goal remains to convey content and skills, rather than the blind subscription to an educational philosophy (no matter how well intentioned it is and no matter how well it may work for others), these techniques can greatly enhance teaching. So, if you are interested in doing something new, check out the next sections before moving on to the class preparations. If not, then most likely you know what you are looking for, and I hope you find it in the preps.

## E.1 Some Active/Cooperative Learning Techniques

*The best answer to the question, "What is the most effective method of teaching?" is that it depends on the goal, the student, the content, and the teacher. But the next best answer is, "Students teaching other students."*

*(Wilbert McKeachie)*

Facilitating active learning is a challenging task. There is no one method that will engage all students and keep them active. This is because students' preferred learning styles can differ widely even in a class as small as 20 students. For a classification of various learning styles, see the Learning Styles Inventory at

<http://www.engr.ncsu.edu/learningstyles/ilsweb.html>

On the positive side, although there is no one method guaranteed to keep all students engaged, there also is no one method that turns all students off. Advocacy of active learning is sometimes mistaken as "lecture bashing." This is either a misperception or, if lecturing actually is belittled, highly inappropriate. For example, lecturing and verbal communication are a perfect fit for verbal learners. Moreover, lecturing is very effective for transmitting the same information to a large group of people. Rich Felder, a convincing and strong advocate of active learning, said that he devotes about 15% of his class time to active learning activities. So *active learning techniques supplement, but do not replace, lectures.*

Active learning techniques are supposed to make lecturing more effective by keeping students engaged or by preparing them for the next segment of a lecture. When placed appropriately, they can enhance the value of a lecture. The descriptions presented here list some of the main active learning technique advocated in education. I have used most of them and I know that they can and do work. But it is virtually certain that there are other ways that will fit better with other individuals' teaching styles. Please consider this section as food for thought to devise your own approaches. The main goal is to keep students' minds engaged throughout the class period. This is what active learning means – it is not tied to a particular style of instruction. Another goal is to improve students' communication ability, because it is virtually assured that they will work in teams as professionals. Here is where cooperative learning is important, because communication with each other starts the training of abilities needed later as professionals.

The list of techniques must be started with a warning. Anyone who tries to implement all of these techniques at once most likely will not meet with success. Modifying your teaching style is a lengthy process, and there is no one sure path to success. Moreover, too many different techniques at once simply distract from the content. The standard recommendation is to try one or two techniques in a term and decide if they work or not. If they do, continue their use. If not, check if a modification will do, or abandon the technique for now.

- **Think-Pair-Share.** Two partners solve the same problem independently. After they are done, they compare methods and results. Differences are explained and, if they

stem from mistakes, eliminated.

Trains oral communication, the justification of steps that differ from someone else's, the tracking of mistakes, and the ability to accept different solution methods.

- **Peer Editing.** Similar to Think-Pair-Share, two partners solve the same problem independently. After they are done, they trade their solutions and edit/grade the partner's work.

Trains written communication and the spotting of mistakes. Also shows that if peers have trouble following a solution, then the instructor most likely will, too.

- **Thinking Aloud Paired Problem Solving (TAPPS).** One partner is the "talker", the other is the "listener" in this activity. The talker's job is to solve a problem as perfectly as possible, while explaining what is done in every step. Essentially the talker gives a mini-lecture to the listener. The listener's job is to listen and spot mistakes. As mistakes occur or the talker gets stuck, the listener is *not* supposed to take over, even if the listener knows the solution. Instead, the listener is supposed to ask leading questions such as "Are you sure your addition is right?" or "Why don't you try rationalizing the denominator?" to put the talker back on the right track.

Roles should be reversed regularly to give both partners the talker and the listener experience.

Trains oral communication and listening skills. Based on the insight that the highest rates of content retention are achieved in tasks in which one does something while explaining it (for some details, see R. B. Lewis (1991), *Creative Teaching and Learning in a Statics Class*, Engineering Education, 15-19).

- **Jigsaw.** This activity works for groups of various sizes. Each group member becomes a "specialist" who is assigned a different task to complete. This can be done alone, or, in a full jigsaw, by joining the other groups' specialists with the same task. When forming groups of specialists with the same task, usually some training is involved before the task can be completed.

Upon completion of tasks, in a full jigsaw the original groups are re-formed, otherwise, students just agree to start reporting, and each group member explains his/her task to the remaining group members.

Trains oral communication and listening skills. To enforce individual accountability for all parts of the jigsaw, a test could include questions on each task.

The obvious danger with this technique is of course that everyone will learn their specialty well and not know much about everyone else's specialty. Moreover there is the temptation for the instructor to cram too many specialties into an individual class period.

- **Enhanced lecture.** This idea only involves the instructor. Stop for a minute or two halfway through a longer lecture to allow students to re-focus. The author has also heard of colleagues make students do jumping jacks when they note that attention is wavering.

It sounds a bit silly and it is hard to keep yourself silent for a whole minute with nothing to do, but it re-focuses the class. It is important to make sure that during such a short break, people do not reach for distractions. We are somewhat conditioned to think that we always have to *do* something. A quick peek at a newspaper during a short break would negate the break's effect. On the other hand, going to the bathroom should be o.k.

Combinations of the above techniques can also be effective. For example, after a certain time during an activity, students find their own way to "enhance" the lecture. Sometimes they are stuck and the break is forced, sometimes they are done, sometimes they simply need a break. This is not a bad thing. As much as we all want everyone to be fully concentrated for 50 minutes, 75 minutes, 90 minutes, 110 minutes (that's the longest class

period I have taught, and, yes, it is possible to concentrate that long), continued effort does lead to fatigue, which can only be resolved by rest. Also, TAPPS activities sometimes turn into group problem solving if both partners get stuck. There is nothing wrong with such modifications as long as students remain actively engaged with the content.

Aside from students actively engaging the content, a big benefit of these techniques is that the instructor gets to observe students at work. These observations as well as questions asked by students in individual discussions can be used to guide further presentations. Experienced instructors can often predict what parts of a class will be the most troublesome for students. If the instructor teaches a class for the first time, the feedback through observations can make up for lack of experience. Moreover, even in classes that I have taught frequently, observations can reveal student difficulties that I was previously unaware of.

**Module 1: Modeling with Differential Equations.**

General introduction could also draw upon Before Module 3, Before Module 6 and Before Module 7. I typically do this in the first class meeting and this section of the guide is set up accordingly.

**Suggested Time.** 0.5-1 class period.

**Lecture/Presentation.**

- What does it mean for a function to be a solution of a differential equation? What happens if we try to plug in a function that is not a solution?
- Introduce the informal and the formal definition of a differential equation, an initial value problem and their solutions. Plus note that the number of constants is (usually) equal to the highest occurring derivative.  
One reason we do not need more than  $n$  initial conditions is that once we have  $n$  conditions, we could solve for the highest order derivative.
- Is  $y(x) = 3x + 1$  a solution of  $x^2y'' + xy' - 4y = 0$ ? Is  $y(x) = x^2$  a solution? (Or set up other easy-to-verify equations.)  
Present some differential equations and their solutions. Verify that the function is a solution by plugging in. Students should do at least one like that on their own. Calm any possible insecurities about finding solutions. (At this stage, solutions “come out of thin air”.)
- Solutions can be ambiguous.
- The class will show only a few specific solution methods. This exposure is to provide tools for other classes as well as to reenforce calculus.
- Recall that every integral is a differential equation of the form  $y' = f(x)$ .
- Differential Equations occur naturally in physics and other sciences; they describe physical phenomena with high accuracy.  
Sometimes the comment that “That’s not my major.” comes up. To alleviate problems in this direction, note that it was attempted to connect to every major. Beyond the attempts, a connection might exist, but not be visible yet.  
Another possibility is to connect specific examples with specific majors and trying to have each major connected to at least one example. Knowing your students’ background builds a lot of good will.
- Good introductory modeling problems are hard to find, so this may be the only available practice in modeling for a while.  
It may be necessary to alleviate the fear that the whole class will be as hard as the derivations of the models. It won’t, but models at the start give us a context.
- Start with Newton’s law of cooling and/or the differential equation for the spring-mass-system.
- Newton’s law of cooling: Should the  $k$  in Newton’s law of cooling be positive or negative?
- Spring-mass-system:
  - Do the spring-mass-system slowly, first without friction and outside force. Then introduce these terms also.  
Only present selected examples. The point is to give a large variety of students an example that connects to their field, not blanket coverage of all fields.
  - Examples of spring mass systems: shock absorbers in cars, trains, springs in chairs, spring scales (fish scales?), pressure gauges (?), watch springs (historically interesting)

- The spring force is the only purely intrinsic force of the spring-mass-system.
  - Friction is an intrinsic as well as an extrinsic force: There is friction through the deformation of the spring (intrinsic: essentially “spring atoms against spring atoms”) and friction through air drag etc. (extrinsic: depends on density and viscosity of the surrounding medium)
  - Gravity is extrinsic to the system.
  - Why are the signs set up the way they are in the equation of a spring-mass-system?
- Could also set up a mixing problem like the ones that will be shown in Section 2.1.
  - Circuit examples: Will become relevant and more understandable in later circuit theory classes.
  - Simple pendulum: Why is it acceptable to replace  $\sin(\Theta)$  with  $\Theta$  in the differential equation of the pendulum?
  - For demonstration of LC circuits, the LC demo at [http://www.valdosta.edu/~cbarnbau/math\\_demos\\_folder/](http://www.valdosta.edu/~cbarnbau/math_demos_folder/) is very useful.

### Group Work/Examples.

- Activity on p. 331.
- Activity on p. 333.
- Check if the given function solves the differential equation.
  1.  $y' = 3y$ , one solution is  $y(x) = e^{3x}$
  2.  $y' = 3y$ ,  $y(0) = 4$ ,  $y(x) = e^{3x}$  is not a solution of the initial value problem, but  $y(x) = 4e^{3x}$  is.
  3.  $x^2y'' + 2xy' + 4y = 0$ , solutions  $y(x) = x^{-3}$ ,  $y(x) = x^2$ .
  4.  $y'' - 2y' + 2y = 0$   $y(x) = e^x \sin(x)$   
(This is an oscillation with rising amplitude. Could make a point of asking how this fits with spring-mass-systems.)
  5.  $y'' + 5y' - 3y = 2e^x$ , solution  $y(x) = Ae^x$ , how do we choose  $A$ ?
- For a specific cup of tea or spring mass system, is it enough to know the differential equation that models it? Lead into initial value problems.

### Section 1.5. Loaded Horizontal Beams (and Introduction to Boundary Value Problems).

**Suggested Time.** 0.5 class periods.

#### Lecture/Presentation.

- To introduce BVPs note that there is a fundamentally different way to picture the independent variable(s)
  1. IVP: independent variable is time
  2. BVP: independent variable is space
  3. (If the section is used as a bridge between IVPs and PDEs) PDE: independent variables represent space and time
- In a boundary value problem we may also need to determine the possibilities for the coefficients of the DE.
- Explain BVP for embedded beams,
- Explain the boundary conditions for embedded beams,
- Derive the BVP for a hanging cable (lengthy), show data to support the computation

#### Group Work/Examples.

- Show that the given function solves the given boundary value problem.
  1. BVP:  $y'' + y = 0$ ,  $y(0) = y(\pi) = 0$ ; function:  $f(x) = \sin(x)$
  2. BVP:  $y'' + y = 0$ ,  $y(0) = y(\pi) = 0$ ; function:  $f(x) = 4 \sin(x)$
  3. BVP:  $y'' + y = 0$ ,  $y(0) = y(\pi) = 0$ ; function:  $f(x) = \sin(2x)$
  4. BVP:  $y'''' = -24$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $y(1) = -1$ ,  $y'(1) = -8$ ; function:  $f(x) = -x^4 - 4x^3 + 4x^2$
- Activity on p. 315

#### Reading Quiz.

1. The independent variable in a boundary value problem is often
  - (a) Time
  - (b) Space
  - (c) Temperature
  - (d) Frequency
2. The differential equation that describes how a loaded beam bends is
  - (a)  $y'' - \mu y = 0$ , where  $\mu$  is a constant,
  - (b)  $y'' - \mu y = c$ , where  $\mu$  and  $c$  are constants,
  - (c)  $EI \frac{d^4}{dx^4} y = w$ , where  $E$ ,  $I$  and  $w$  are constants,
  - (d)  $EI \frac{d^4}{dx^4} y = w(x)$ , where  $E$  and  $I$  are constants,
3. The boundary value problem  $y'' - \mu^2 y = 0$ ,  $y(-a) = y(a) = h + \frac{1}{\mu}$ ,  $y(0) = \frac{1}{\mu}$  for the suspended, unloaded cable will be
  - (a) Solvable for some combinations of  $\mu$ ,  $a$  and  $h$ , but not for all,
  - (b) Solvable for all combinations of  $\mu$ ,  $a$  and  $h$ ,
  - (c) Unsolvable for all combinations of  $\mu$ ,  $a$  and  $h$ ,
  - (d) This is not the boundary value problem for a suspended cable.

### Section 2.1: Separable ODEs.

**Suggested Time.** 1-1.5 classes.

#### Lecture/Presentation.

- First solve a real easy separable equation, say,  $y' = yx$ .
- Then solve  $y' = yx \cos(x^2)$ ,  $y(0) = 1$ .
- Explanation how to find  $c$  should be quick.
- Stress that we need to be able to find integrals in order to solve separable differential equations.
- Solve  $y' = e^{x^2+y}$ ,  $y(3) = 2$ . Explain how we use the fundamental theorem of calculus to obtain a symbolic solution.
- General comment: When integrating  $\frac{1}{x}$  we use  $\ln|x|$ , but when solving  $e^y = \dots$  we use  $y = \ln(\dots)$ .
- Explain a mixing problem.

A water tank contains 100l of brine with  $50\frac{g}{l}$  of salt. Fresh water enters the tank at a rate of  $3\frac{l}{min}$  and mixture exits the tank at the same rate. Assuming perfect mixing, how long will it take until the concentration of salt is less than  $1\frac{g}{l}$ ?

Track the total amount of salt  $S(t)$ .  $S'(t) = -\frac{3}{100}S(t)$ ,  $S(0) = 5,000g$

Obviously a “made up” problem, but it occurs in reality in the form of pollution in lakes that are fed by freshwater streams and lose water through an outlet.

- Note that, even though the section also gives a quick run through the major integration techniques and the Fundamental Theorem of Calculus, memorization of too much detail is counterproductive, and that a small number of examples ultimately is (must be?) enough.
- Note that double checking assures you that you got it right.
- For really ugly equations and solutions, use a CAS to double check.

#### Group Work/Examples.

- Activity “Separable Differential Equations” on page 305 can be used to structure the class as follows.
  - Let students do the first part of the activity.  
Careful when solving  $e^{-y} = \dots$ . Use  $\ln(\cdot)$ , not  $\ln|\cdot|$ .
  - Introduce separable differential equations, using some of the ideas from the Lecture/Presentation part.
  - Finish with the “Practice Problems” on the activity.
- Solve: In a room kept at 68F a cup of hot tea (150F) is left to cool. Assuming  $k=-.2$ , when does the tea have a temperature of 108F?  
Let students discuss what is needed for the solution: differential equation, initial value problem, solution of the initial value problem, computation of the time.  
Stress how modeling and solution method go hand-in-hand.



**Section 2.2: Linear First Order Differential Equations.****Suggested Time.** 0.5 class period.

For the remainder of the instructor's guide, examples will be listed under Group Work/Examples.

**Lecture/Presentation.**

- Start emphasizing that memorization of too much detail is counterproductive and that a small number of examples is (must be?) enough. Students who rely on excessive memorization or need large numbers of examples may be trying to cover calculus weaknesses. It's better to address the weaknesses than to mask them.
- The only parts to remember are the form of the equation,  $y' + p(x)y = q(x)$ , and the solution method: Multiply with the integrating factor  $\mu(x) = e^{\int p(x) dx}$ .
- Quick explanation why the solution method works. (A proof that is just not called a proof.)
- Remind students that double checking assures you that you got it right.

**Group Work/Examples.**

- Activity on page 306.
- Solve the differential equation  $y' + \frac{2}{x}y = e^x$ .
- Solve the initial value problem  $y' + y = \cos(e^x)$ ,  $y(0) = 3$ .
- Solve the initial value problem  $y' + \cos(x)y = \cos(x)$ ,  $y(0) = 5$ .  
(This one is also separable and might be recognized as such.)
- Solve the initial value problem  $y' + y = \sin(x)$ ,  $y(0) = 2$ .

**Section 2.3: Bernoulli Equations.**

**Suggested Time.** 0.5 class period.

**Lecture/Presentation.**

- It's worth repeating that memorization of too much detail is counterproductive and that a small number of examples is (must be?) enough.
- The only parts to remember are the form of the equation,  $y' + p(x)y = q(x)y^n$  and the solution method: Substitute  $y = v^{\frac{1}{1-n}}$ .
- Quick explanation why the solution method works. (A proof that is just not called a proof.)
- Remind students that double checking assures you that you got it right.

**Group Work/Examples.**

- Activity on page 307.
- Solve the initial value problem  $y' + 2y = y^2$ ,  $y(0) = -2$ .
- Solve the initial value problem  $y' + \frac{3}{x}y = \frac{1}{y}$ ,  $y(1) = 3$ .
- Solve the initial value problem  $y' + \frac{2}{x}y = e^x \sqrt{y}$ ,  $y(1) = 4$ .
- Solve the initial value problem  $y' + \tan(x)y = y^3$ ,  $y(0) = 2$ .

**Section 2.4: Homogeneous Equations.**

**Suggested Time.** 0.5 class period.

**Lecture/Presentation.**

- It's worth repeating that memorization of too much detail is counterproductive and that a small number of examples is (must be?) enough.
- The only parts to remember are the form of the equation,  $y = f\left(\frac{y}{x}\right)$ , and the solution method: Substitute  $v = \frac{y}{x}$ , use  $y = vx$  to get  $y'$ .
- Quick explanation why the solution method works. (A proof that is just not called a proof.)
- Remind students that double checking assures you that you got it right.

**Group Work/Examples.**

- Activity on page 308.
- Solve the initial value problem  $y' = \frac{y^2}{x^2} + \frac{y}{x} + 1$ ,  $y(1) = 1$ .
- Solve the initial value problem  $y' = \tan\left(\frac{y}{x}\right) + \frac{y}{x}$ ,  $y(1) = 2$ .
- Solve the initial value problem  $y' = \frac{1}{\cos\left(\frac{y}{x}\right)} + \frac{y}{x}$ ,  $y(1) = 1$ .
- Solve the initial value problem  $y' = \frac{y^2}{x^2}$ ,  $y(1) = -2$ .

### Section 2.5: Exact Differential Equations.

**Suggested Time.** 0.5 class period.

#### Lecture/Presentation.

- It's worth repeating that memorization of too much detail is counterproductive and that a small number of examples is (must be?) enough.
- The only parts to remember are the form of the equation,  $M(x, y) + N(x, y)y' = 0$  with  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  and the solution method: Find a potential function  $f$  so that  $\nabla f = \begin{pmatrix} M \\ N \end{pmatrix}$ .
- Quick explanation why the solution method works. (A proof that is just not called a proof.)
- Note that if the equation cannot be solved for  $y$ , then it should be left in implicit form.
- Remind students that double checking assures you that you got it right.

#### Group Work/Examples.

- Activity on page 309.
- Determine if the differential equation  $4x^2 - 2y + 2y' = 0$  is exact.
- Determine if the differential equation  $-\sin(x)\sin(y) + \cos(x)\cos(y)y' = 0$  is exact.
- Solve the initial value problem  $x^2 + y^2y' = 0$ ,  $y(0) = 2$ .
- Solve the initial value problem  $\cos(y) - (x\sin(y) + 2)y' = 0$ ,  $y(1) = 3$ , if the equation is exact.
- Solve the initial value problem  $x - xy' = 0$ ,  $y(1) = 3$ , if the equation is exact. (Can we solve the problem in another way?)
- Solve the initial value problem  $3x^2 + y + (x + 2y)y' = 0$ ,  $y(1) = 3$ , if the equation is exact.
- Solve the initial value problem  $xe^{xy} + ye^{xy}y' = 0$ ,  $y(1) = 3$ , if the equation is exact.

### Section 2.6: Scoring Rubric/Checklist for First Order Differential Equations Projects.

The scoring rubric below is approximately what will be used to score the projects in this section. It can be used as a check sheet to determine completeness of the project. While completeness according to this rubric does not guarantee a perfect score, incompleteness certainly guarantees an imperfect score. For details, check with your instructor, who is the final authority in all matters pertaining to your class.

“Live” computation means that, once the problem is entered, all other steps are computed without adjustment by the programmer.

To be submitted for each project:

- One copy of the program (as inputs, use the values of task 1),
- In a separate document, solutions for all tasks (not the programs themselves),
- For tasks that could not be completed, a written explanation why the program did not work.

Assessment item	percentage	check/score
Inputs encoded correctly	10%	
Intermediate and final outputs computed “live”	10%	
Intermediate functions correct	10%	
General solution correct	10%	
Constant $c_0$ correct	5%	
Graph of solution	10%	
Check if solution solves the differential equation	10%	
Check if solution has the right initial value	10%	
Correct solutions to the tasks	10%	
Fixed problems with program in tasks where the program got stuck (if applicable)	10%	
Grammar/Style/Effective presentation of computations	5%	

**Section 2.8: Review of First Order Differential Equations.**

**Suggested Time.** 0.5 class period.

**Lecture/Presentation.**

- Students should read Section 2.7 “How to Review and Remember” very carefully.
- Here’s where memorization of *the right formulas* pays off. After identifying the type of the equation (types need to be memorized), we solve the equation (solution methods need to be memorized).

**Group Work/Examples.**

- Solve the initial value problem  $xy' + y = x \sin(x)$ ,  $y\left(\frac{\pi}{2}\right) = 0$  (linear).
- Solve the initial value problem  $y' - \frac{y^3}{x^3} - \frac{y}{x} = 0$ ,  $y(1) = 1$  (homogeneous).
- Solve the initial value problem  $\cos(y)y' = x$ ,  $y(0) = \frac{\pi}{2}$  (separable).
- Solve the initial value problem  $y' + y + e^x \sqrt[3]{y} = 0$ ,  $y(0) = 1$  (Bernoulli).
- Solve the initial value problem  $y^3 + 2x + (3xy^2 + y^3)y' = 0$ ,  $y(0) = 3$  (exact).

### Section 3.1: Homogeneous Linear Differential Equations with Constant Coefficient

**Suggested Time.** 0.5-1 class periods.

#### Lecture/Presentation.

- Introduce linear differential equations, recall spring-mass-systems and LRC circuits.
- Reiterate the importance of linear differential equations in practice. Because of this importance, we are providing a tool for applications across all courses.
- The theoretical results presented in Module 5 provide the foundation for the solution methods we are to develop. They guarantee that we find all solutions.
- Central Theme: Using  $e^{\lambda t}$  to solve differential equations. Talk about the importance of recognizing a pattern. Compare working with a parameter  $\lambda$  that we can adjust in  $e^{\lambda t}$  to the use of parameters in partial fraction decompositions.
- Mention the definition of  $e^{\lambda t}$  with complex  $\lambda$ . The complex exponential function is just the tool we need.
- Derivative of  $e^{\lambda t}$  for complex  $\lambda$ .
- **Motivation of the differential equations via spring-mass systems, observations that solutions are describing real life phenomena.** This is probably the biggest tie in. Motivate the equations on the activity as differential equations of spring mass systems with various springs and friction setups.  
 Could use Garrett Heath's Harmonic Oscillator applet, see [12], with fixed mass and spring constant to show what happens as friction changes from large to small.
- For demonstration of LC circuits, the LC demo at [http://www.valdosta.edu/~cbarnbau/math\\_demos\\_folder/](http://www.valdosta.edu/~cbarnbau/math_demos_folder/) is very nice.  
 There is a pendulum demo, too.
- Critical damping is what is wanted in a system that is to come to rest as fast as possible without oscillating (shock absorbers).
- Show an experiment as indicated in Experimental Verification 3.14.
- Always remember: To check if a differential equation is solved correctly, simply plug in your solution.

#### Group Work/Examples.

- Activity "Solving homogeneous constant coefficient differential equations" on page 311 can be used to guide the class.
- Some more problems on spring mass systems similar to the exercises.
- What is the difference between a homogeneous and an inhomogeneous linear differential equation?
- What is the interpretation of the coefficients and the inhomogeneity in terms of a spring mass system?
- What type of a spring mass system is described by  $2y'' + 10y' + 3y = 0$ ?
- Why does  $y'' - y' + 2y = 0$  not represent a "real" spring mass system?  
 Also determine which of the three could not represent a real spring-mass-system.

**Section 3.2: Solving Initial and Boundary Value Problems**

**Suggested Time.** 0.5 class periods. The only new content is the solution of initial value problems, which can be weaved into coverage of the previous section, if necessary.

**Lecture/Presentation.**

- Use an example to show how to solve an initial value problem for a second order equation.

**Group Work/Examples.**

- Solve the initial value problem
  1.  $2y'' + 5y' + 2y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 4$
  2.  $2y'' - 12y' + 18y = 0$ ,  $y(0) = 3$ ,  $y'(0) = 1$
  3.  $y'' + 4y' + 13y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 1$



### Section 3.3. Designing Oscillating Systems

**Suggested Time.** 1 class period

**Lecture/Presentation.**

- The need to design mechanical and electrical systems that oscillate or the prevention of oscillations in such a system.
- Examples: Shock absorbers, radio senders and receivers, processor clock (even though LRC circuits are not used for that any more)

Reminder: When designing examples in which the system achieves a desired frequency by adjusting the friction, make sure that the natural frequency factor  $\sqrt{\frac{k}{m}}$  is larger than the needed frequency factor. Friction will only slow down the oscillation, so certain frequencies are out of reach.

**Group Work/Examples.**

- For a spring-mass-system with  $m = 400kg$  and  $k = 10,000\frac{N}{m}$  find the friction coefficient that makes the system critically damped.
- For an LC circuit with  $C = 10nF = 10^{-8}F$ , find the inductance  $L$  needed to have a frequency of  $10MHz = 10^9Hz$

### Section 3.4: The Method of Undetermined Coefficients

**Suggested Time.** 1 class period.

#### Lecture/Presentation.

- Explain how even systems that do not naturally oscillate can be made to oscillate with a sufficiently strong driving term. <http://www.falstad.com/diffeq/> is useful for demonstrations.
- Explain that the general solution of the inhomogeneous equation is one particular solution of the inhomogeneous equation plus all solutions of the homogeneous equation,
- Undetermined coefficients only works for equations with constant coefficients and if the derivatives of the inhomogeneity exhibit repeating patterns,
- Could list all the inhomogeneities for which undetermined coefficients works and make a table that says what to do. Setting up the table is educational, memorizing it would be a *terrible* waste of memory.
- Lead in with a sample problem with an easy right hand side, say  $y'' + 4y' + 5y = 3x^2$  (or maybe an exponential right hand side).
- Note that when the inhomogeneity is a solution of the homogeneous equation, then we must multiply it with the appropriate power of  $x$  and use this function alone in the pattern.
- Emphasize the need for a repeating pattern in the derivatives of the inhomogeneity.

#### Group Work/Examples.

- Activity on page 313.
- Let students make a table with possible right sides and corresponding setups. This should be seen as a formalization of the process of setting up the method of undetermined coefficients.
- Why does the problem with the inhomogeneity being the tangent in the sample problems in the activity not work when we try undetermined coefficients?
- To show what happens when the inhomogeneity is a solution of the homogeneous equation
  1.  $y'' + 4y' + 3y = e^{-t}$ ,  $y(0) = 0$ ,  $y'(0) = 0$
  2.  $y'' + 4y = \sin(2t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$   
(This one can be used to talk about resonance.)

#### Notable Homework Problems.

- The table developed next to Solution Method 3.27 and finished in the homework problems can be helpful to students to find the right setup.

**Section 3.5: Variation of Parameters**

**Suggested Time.** 1 class period. For uglier integrals, a CAS would be quite helpful.

**Lecture/Presentation.**

- Derive the formula for Variation of Parameters.
- Thanks to our theoretical preparation, we only need one particular solution for the inhomogeneous equation.
- Variation of Parameters works for all linear differential equations for which the homogeneous solution is known.
- It should be established that the formula is plug and chug. If you can substitute functions into a formula and solve the resulting integrals, it works.
- The main value of the formula is that it always gives a solution, even when the method of undetermined coefficients does not. The integrals can turn out to be pretty ugly, though, and they may be left as definite integrals rather than solved explicitly.

**Group Work/Examples.**

- Why is the method called Variation of Parameters?
- Find the general solution of the differential equation  $y'' + 3y' + 2y = e^{-x}$ . (Integrals are as simple as possible.)
- Find the general solution of the differential equation  $y'' + 4y' + 3y = \sin(x)$ . (Could make the right side equal to  $x$  for easier integrals.)
- Could use an example from the Method of Undetermined Coefficients and compare the methods' pros and cons.

**Section 3.6: Cauchy-Euler Equations**

**Suggested Time.** 0.5 class periods.

**Lecture/Presentation.**

- Definition of Cauchy-Euler equations.
- Explain why the substitution  $y = x^r$  is natural.
- Substitution that reduces Cauchy-Euler equations to constant coefficient equations.

**Group Work/Examples.**

- Find the general solution of the differential equation.
  - $x^2y'' - 4xy' + 4y = 0$
  - $x^2y'' - xy' + y = 0$
  - $x^2y'' - 3xy' + 8y = 0$
- Find the general solution of the differential equation  $x^2y'' - 5xy' + 5y = x^2$ . (Use Variation of Parameters for the particular solution of the inhomogeneous equation.)

### Section 3.7: Some Results on Boundary Value Problems

**Suggested Time.** 0.5 class periods.

Make an activity for the solution of the BVP with the sine and cosine solutions?

#### Lecture/Presentation.

- Show how Propositions 3.41, 3.44 and 3.45 demonstrate the interaction between the conditions and the coefficients.

Demonstrate Proposition 3.45 with an animation of sine functions with increasing frequencies.

- Propositions 3.41, 3.44 and 3.45 are stated very generally because we will use them for reference purposes.

#### Group Work/Examples.

- Activity on p. 315
- Activity on p. 316
- Activity on p. 323 can be used to show that Laplace transforms are cumbersome for boundary value problems. They should be used when there is no other way and they should be avoided otherwise. For example, a point load on a beam would be modeled with the Dirac delta function. Laplace transforms are unavoidable here.
- They find general solutions to DEs, then BVP is solved by the whole class.
- Solve the BVP  $y'''' = -x$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $y''(1) = 0$ ,  $y'''(1) = 0$ . This one can be interpreted as a cantilever beam.

Use two methods (if both are available): Laplace transforms and direct computation.

$$\text{Solution. } y(x) = -\frac{1}{120}x^5 + \frac{1}{12}x^3 - \frac{1}{6}x^2$$

Can also discuss maximum bending moment, which occurs where  $y''$  is maximal. Calculus gives that the second derivative has maxima at  $\pm 2$ , neither of which is in our domain. Since the second derivative is decreasing on  $[0, 1]$  we get the maximum bending moment to occur at the embedding at 0.

- Solve the boundary value problem  $y'' + y = 0$ ,  $y(0) = 0$ ,  $y(2\pi) = 0$ .  
Use Laplace transforms or  $e^{\lambda x}$  (if both are available). Decide which is faster.
- Solve the boundary value problem  $y'' - 4y = 0$ ,  $y(2) = y(-2) = \frac{e^4 + e^{-4}}{2}$ .
- Attempt to solve the boundary value problem  $y'' - y = 0$ ,  $y(-\ln(2)) = y(\ln(2)) = 2$ ,  $y(0) = 1$ . (Example of an unsolvable boundary value problem.)
- Solve the boundary value problem  $y'' + \mu^2 y = 0$ ,  $y(0) = y(\pi) = 0$ . (Example of an underdetermined boundary value problem.) Can use Activity on page 316.  
(Demonstrate that there may be considerable freedom in when solving boundary value problems with an animation that graphs sine functions with increasing frequency factors, see video for BVP of oscillating string. Stop the animation whenever the boundary condition is satisfied.)
- Project 3.8.4 "Catenaries vs. Quadratics" can be assigned after this class.

**Notable Homework Problems.**

- Exercise 13 makes the student solve a boundary value problem that we have solved in the interval  $[0, \pi]$  first on the interval  $[0, b]$  (Exercise 13a) and then on the interval  $[a, b]$  (Exercise 13b). While 13b is maybe unnecessarily tedious, the scaling in 13a is simple and the problem arises frequently in engineering and science.

The translation from Proposition 3.45 to Exercise 13a is simple, but it should be valuable training for students to become accustomed to working with parameters.

### Section 3.8.4. Catenaries vs. Quadratics

For part 9 the author does not believe it is possible to get a difference in length that exceeds 10%.

#### Scoring Rubric for Project 3.8.4.

The scoring rubric below is approximately what will be used to score the project. It can be used as a check sheet to determine if all points that need to be made were made. While completeness according to this rubric does not guarantee a perfect score, incompleteness certainly guarantees an imperfect score.

Assessment item	percentage	check/score
Computation of the quadratic (coefficients $a$ , $b$ and $c$ )	5%	
Proof that quadratics do not satisfy the differential equation for a hanging cable	5%	
Computation of the catenary ( $d$ and $\mu$ )	10%	
Graph of catenary and quadratic	5%	
Computation of lengths and associated error	10%	
Explanation regarding small error in length	5%	
Computation of maximum vertical difference and associated error	10%	
Explanation regarding small error in vertical distance	5%	
Correct solution for all data sets	10%	
Explanation of limitation to $[-L, L]$	5%	
Comparison of functions outside $[-L, L]$	5%	
Realism (or not) of $L = h$	5%	
Values for which errors exceed 10% (if possible)	10%	
Grammar and Style (including data presentation) (A single typo will already reduce this credit to 5%)	10%	

**Section 4.1: Direction Fields**

**Suggested Time.** 0.5 class periods.

**Lecture/Presentation.**

- Section could be used as a lead in to Euler's method and other numerical methods.
- Direction fields are a way to visualize the effect of an explicit first order differential equation  $y' = f(x, y)$ . They may have come up in the treatment of antiderivatives.
- The idea is that the function  $f(x, y)$  prescribes a slope at every point in the plane. (Thus we draw an arrow of unit length in the right direction at every point.)
- Can liken the idea to a description of a wind tunnel (the solution to an IVP would be the trace of a source leaking red dye into the stream) and give the cross-connection to vector fields.
- Explain equilibrium solutions for autonomous differential equations.

**Group Work/Examples.**

- Not sure how much sketching a direction field would do. It would certainly take a long time.
- Activity on page 317.  
Could expand like this: Start with  $y(0) = 1$  and go lower in steps of  $\frac{1}{20} = 0.05$ . Try to determine for what initial value we start drifting away from  $y(2) \approx 2$ .
- Find the equilibrium solutions of the differential equation  $y' = y(y - 1)(y - 2)$  and classify them as stable or unstable.



**Section 4.2. Euler's Method**

**Suggested Time.** 0.5 class periods.

**Lecture/Presentation.**

- Euler's method can be presented as a refinement or formalization of the sketching of solution curves in direction fields.

In this case it is instructive to sketch an execution of Euler's method into a direction field that is projected onto a white board.

- Present the formulas, implement method in EXCEL.

**Group Work/Examples.**

- Apply Euler's method with various step lengths to  $y' = x^2 - (2(y - 1))^2$  on  $[0, 2]$ .
- Is it preferable to have a symbolic solution or a numerical approximation?
- How do we reduce the error for Euler's method?

**Section 4.3. Runge-Kutta Methods**

**Suggested Time.** 0.5 – 1 class periods.

**Lecture/Presentation.**

- Present the methods as refinements of Euler's method.
- Implement method in EXCEL.

**Group Work/Examples.**

- Apply the improved Euler method and the Runge-Kutta method with various step lengths to  $y' = x^2 - (2(y - 1))^2$  on  $[0, 2]$ .
- Compare the performance of Euler, improved Euler and Runge-Kutta.

**Section 4.4. Finite Difference Methods for Second Order Boundary Value Problems**

**Suggested Time.** 0.5 – 1 class periods.

**Lecture/Presentation.**

- Derive the idea for finite difference schemes.
- Solve all systems of equations that arise with a CAS.

**Group Work/Examples.**

- Use a finite difference scheme to solve the boundary value problem  $y'' + y = 0$ ,  $y(0) = 1$ ,  $y(\pi) = -1$ .  
Compare the approximation with the exact solution  $y(x) = \cos(x)$ .

### Sections 5.1-5.5: Existence Theorems and Linear Independence

Central theme: Form of the general solution of a linear differential equation. Linear algebra as the key concept in formulating the result.

**Suggested Time.** 1 class period.

#### Lecture/Presentation.

- Show that distinct solutions of linear higher order differential equations can cross each other. They just can't be equal in all first  $n$  derivatives at any point.
- Show with example 5.16 or with something like  $e^x$  and  $e^{x+1}$  that the informal definition of the general solution in Convention 1.9 is problematic.
- Explain that the Existence and Uniqueness Theorem (Theorem 5.2) shows that solutions to linear differential equations somehow are related to vectors.
- Explain that the Superposition Principle (Theorem 5.4) shows that for homogeneous equations the analogy also includes algebraic properties.
- Note that inhomogeneous equations are handled by finding a particular solution and all homogeneous solutions.
- Introduce the concept of linear independence with two sets of three vectors. Contrast an independent set with a dependent set.
- Linear independence is the key concept that allows us to say when all members of a set of functions or vectors are “truly different” from each other.

**This is probably the most work. Linear independence is a tough concept.**

- Determine if the set of vectors  $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$  is linearly independent.
- Introduce determinants as a seemingly quicker way to show linear dependence or independence. (For more than 3 variables, determinants are not so quick, but for up to 3 variables, they are o.k.)
- Introduce linear independence for functions. Note that we want the same idea of the functions “pointing in different directions”.
- Introduce the Wronskian.
- Prove that  $\{t, e^t, te^t\}$  is a linearly independent set of functions.
- Prove Theorem 5.23 on the general form of the solution of linear differential equation.
- To emphasize the utility of the theory, note that the theory allows us to break linear differential equations into homogeneous (easier) and inhomogeneous (harder) differential equations, and that the theory shows that we won't need to work too much with the inhomogeneous equations.

#### Group Work/Examples.

- Does the method of setting up  $y = e^{\lambda t}$  really give us all solutions? How would we know?
- Activity on linear combinations on page 319.
- Activity on linear independence and dependence on page 320.
- Graphical representation of linear combinations.

- Set of 3 vectors, two are multiples of each other; show that they are dependent, yet there is one that cannot be expressed as a combination of the others.
- The functions  $\sin(x)$  and  $\cos(x)$  both solve  $y'' + y = 0$ . Yet they intersect each other. Doesn't the existence and uniqueness theorem forbid this?
- The functions  $\sin(x)$ ,  $\cos(x)$  and  $\cos\left(x + \frac{\pi}{3}\right)$  all are solutions of the differential equation  $y'' + y = 0$ .

$$y(x) = C_1 \cos(x) + C_2 \sin(x) + C_3 \cos\left(x + \frac{\pi}{3}\right)$$

has one too many constants to be the general solution of the differential equation. What happened? Can you find the general solution of the differential equation?

- Suppose the exponential setup leads us to the conclusion that  $(\lambda - (1+i))^3$  is a factor of the resulting polynomial. What do the solutions that correspond to this factor look like?

### Section 6.1: Introducing the Laplace Transform

**Suggested Time.** 0.5-1 classes.

**Lecture/Presentation.**

- Show the Laplace transform of the derivative as motivation.
- Show how to compute Laplace transforms, inverse Laplace transforms.

$$\begin{aligned} & - \mathcal{L}\{e^{at}\} \\ & - \mathcal{L}\{t\} \\ & - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-5)}\right\} \end{aligned}$$

- Linearity of the Laplace transform and its inverse.
- Partial fraction decompositions will be important throughout the module.
- Use of computer algebra systems for complicated partial fraction decompositions and also for Laplace transforms.
- Apparently there is no simple computational formula for the inverse Laplace transform.
- Definition of exponential order and an example of a function that is not of exponential order, say,  $f(x) = e^{x^2}$ .

**Group Work/Examples.**

- Activity on page 321.
- Work out how to find Laplace transforms on their CAS, compare results with computations in class.
- Can assign the project 6.7.1 “Exact Laplace Transforms” right after this class.
- Can the methods we have learned so far for linear differential equations with constant coefficients handle discontinuous inhomogeneities?
- Why is there no problem with the Laplace transform only acting on  $[0, \infty)$ ? (Because we are only interested in the future behavior.)
- Is it necessary for a function to be exponentially bounded for the Laplace transform to exist? (There are pathological examples that it is not. This is something for theoretically inclined classes.)

## Section 6.2: Solving Differential Equations with Laplace Transforms

**Suggested Time.** 0.5-1 classes.

### Lecture/Presentation.

- The Laplace transform changes differential equations into algebraic equations. The method “transform-solve-transform back” is the standard technique to solve a differential equation with Laplace transforms. (See Solution Method 6.22.)
- If we know enough about the Laplace transform, we will be able to solve constant-coefficient equations with any inhomogeneity.
- To check if a differential equation is solved correctly, we still simply plug in the solution.

### Group Work/Examples.

- Solve the initial value problem.

$$- y'' + 4y' - 12y = 0, y(0) = 1, y'(0) = 3$$

$$\text{Solution: } y(t) = -\frac{1}{8}e^{-6t} + \frac{9}{8}e^{2t}$$

$$- y'' + 3y' + 2y = e^{-3t}, y(0) = 0, y'(0) = 0$$

$$\text{Solution: } y(t) = -e^{-2t} + \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

- In an RLC circuit with  $R = 100\Omega$ ,  $L = 100mH = .1H$ ,  $C = \frac{1}{24840}F \approx 40.2\mu F$ , the capacitor is initially uncharged and a current of  $1A$  is flowing. Find the charge of the capacitor as a function of time. (Partial fraction decomposition gets a little hairy.)

$$\text{Solution: } y(t) = -\frac{1}{80}e^{-540t} + \frac{1}{80}e^{-460t}$$

**Reading Quiz.**

1. There are scribbles in the margin of my book. (Circle all that apply.)
  - (a) The marginal notes model a way of working that I am encouraged to adopt.
  - (b) The bookstore sold me a used book that some fool smeared all over.
  - (c) My book has no scribbles.
  - (d) I don't have a book.
  
2. Which of the following is the definition of the Laplace transform?
  - (a)  $\mathcal{L}\{f\} = \int_{-\infty}^{\infty} f(t)e^{-st} dt$
  - (b)  $\mathcal{L}\{f\} = \int_0^{\infty} f(t)e^{-st} dt$
  - (c)  $\mathcal{L}\{f\} = \int_0^{\infty} f(t)e^{st} dt$
  - (d) None of the above
  
3. Which is the most effective way to find the general solution of a second order constant coefficient linear differential equation with Laplace transforms?
  - (a) This is not possible, Laplace transforms can only be used to solve initial value problems.
  - (b) We set  $y(0) = 1, y'(0) = 0$  to obtain a solution  $y_1$  and we set  $y(0) = 0, y'(0) = 1$  to obtain a solution  $y_2$ . The general solution is  $y(t) = c_1y_1(t) + c_2y_2(t)$ .
  - (c) We transform the equation and keep  $y(0)$  and  $y'(0)$  as symbolic constants throughout the remainder of the computation.
  - (d) We set  $y(0) = 1, y'(0) = 0$  and obtain a solution  $y_1$  of the homogeneous equation (right side equal to zero). Then we set  $y(0) = 0, y'(0) = 1$  to obtain another solution  $y_2$  of the homogeneous equation. If there is an inhomogeneity, we also find a solution  $y_p$  for  $y(0) = y'(0) = 0$  for the inhomogeneous equation. The general solution is  $y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$ .
  
4. The Laplace transform of  $y'(t)$  is
  - (a)  $sY - y(0)$
  - (b)  $s^2Y - sy(0) - y'(0)$
  - (c)  $Y - sy(0)$
  - (d)  $Y^2 - s^2$



### Section 6.3: Systems of Linear Differential Equations

**Suggested Time.** 1 class period. The modeling part is probably harder than solving the systems.

**Lecture/Presentation.**

- Systems of differential equations arise when several components (which are governed by individual differential equations) are interacting.
- To check if a system of differential equations is solved correctly, simply plug in your solution.
- Kirchhoff's laws.
- Newton's second law of motion.
- Solve a simple example of a system of differential equations.

1.

$$\begin{aligned}x' + y' + y &= 0 \\x - y &= e^t \\x(0) = 1, y(0) &= 0\end{aligned}$$

Solution.  $x(t) = \frac{1}{3}e^{-\frac{1}{2}t} + \frac{2}{3}e^t, y(t) = \frac{1}{3}e^{-\frac{1}{2}t} - \frac{1}{3}e^t.$

This one is simple, but careful: Initial conditions are not independent of each other because the second equation is not a differential equation. This should not bother us. Some circuits also give us systems involving equations that are not differential equations.

One task in Project 6.7.3 has the above system with zero initial conditions, which is ill-defined.

2.

$$\begin{aligned}x' + x + y' + y &= 0 \\x' + 4x - y &= 0 \\x(0) = 0, y(0) &= 1\end{aligned}$$

Solution.  $x(t) = -\frac{1}{4}e^{-5t} + \frac{1}{4}e^{-t}, y(t) = \frac{1}{4}e^{-5t} + \frac{3}{4}e^{-t}.$

- Solve a simple circuit example.

A resistor  $R_1$  is in series with another resistor  $R_2$ , which is parallel to the capacitor  $C$  (see Figure 6.15).

$$R_1 = 50\Omega, R_2 = 100\Omega, C = 1mF = \frac{1}{1000}F, E(t) = 3V$$

Initially no currents are flowing and the capacitor is uncharged.

Find the currents  $i_2$  and  $i_3$  and note how a capacitor blocks directed current.

Solution.

$$\begin{aligned}50q' + 150i_3 &= 3 \\1000q - 100i_3 &= 0\end{aligned}$$

$$\begin{aligned}q(t) &= \frac{1}{500} - \frac{1}{500}e^{-30t} \\i_3(t) &= \frac{1}{50} - \frac{1}{50}e^{-30t}\end{aligned}$$

- Solve a simple coupled spring-mass-system.

For example for the system depicted in Figure 6.13, let  $m_1 = m_2 = 1\text{kg}$ ,  $x_1(0) = 1$ , all other initial conditions zero. Then the following combinations  $k_1, k_2$  give “easy” systems. (Need  $k_1^2 + 4k_2^2$  to be a square of an integer. For each pair, an integer multiple will also work.)

$$k_1 = 3\frac{N}{m}, k_2 = 2\frac{N}{m},$$

$$k_1 = 5\frac{N}{m}, k_2 = 6\frac{N}{m},$$

$$k_1 = 7\frac{N}{m}, k_2 = 12\frac{N}{m},$$

$$k_1 = 8\frac{N}{m}, k_2 = 3\frac{N}{m},$$

$$k_1 = 12\frac{N}{m}, k_2 = 8\frac{N}{m}, \text{ (one integer frequency factor)}$$

$$k_1 = 15\frac{N}{m}, k_2 = 4\frac{N}{m},$$

The coupled systems are a good place to show that Heaviside's method will not always work. Setting up the  $4 \times 4$  system for the coefficients is the best approach for these coupled systems.

### Group Work/Examples.

- (In the coupled spring-mass-systems.) Why does the force exerted by spring 2 on mass 1 have the opposite sign from the force exerted by spring 2 on mass 2?
- (As a lead in for what's to come.) What types of external voltages can be put into a circuit and can we Laplace transform them yet?
- Projects 6.7.3 “Linear Systems via Laplace Transforms” or 6.7.4 “Linear Second Order Systems via Laplace Transforms” can be assigned after this class.

### Reading Quiz.

- To check the solution of a system of two differential equations for two functions with initial value problem we do the following.
  - Solutions of systems of differential equations cannot be checked for correctness.
  - Plug in  $x(t)$  for  $x$  and  $y(t)$  for  $y$  in the equations. If both sides are equal the solution is correct.
  - Plug in  $x(t)$  for  $x$  and  $y(t)$  for  $y$  in the equations. If both sides are equal and  $x(t), y(t)$  satisfy the initial condition, the solution is correct.
  - We plot the graphs and see if they look right.
- We already have a method to solve linear constant coefficient differential equations. Why do we need Laplace transforms? Check all that apply.
  - Our previous methods do not allow us to consider discontinuous right sides.
  - Laplace transforms (eventually) turn out to be computationally simpler.
  - We don't need Laplace transforms.
  - Laplace transforms easily generalize to systems of equations.
- Kirchhoff's voltage law states that
  - The sum of all voltages at a node is zero.
  - The sum of all currents at a node is zero.
  - The sum of all voltages in a closed loop is zero.
  - The sum of all currents in a closed loop is zero.

**Section 6.4. Expanding the Transform Table.****Suggested Time.** 1 class period.**Lecture/Presentation.**

- For the rest of this module we will re-enforce what we have learned in the first three sections. The computations to solve differential equations and systems will not change. Neither will the formula for the Laplace Transform.
- Theorems 6.30 and 6.32, which give the Laplace transforms of  $t^n f(t)$  and of  $e^{at} f(t)$ , are motivated by the shapes of solutions we have seen in Module 3.
- Prove Theorems 6.30 and 6.32 (or let students do one proof as group work)
- Find some Laplace transforms.

$$1. \mathcal{L} \left\{ t^2 \sin(t) \right\} = \frac{6s^2 - 2}{(s^2 + 1)^3}$$

$$2. \mathcal{L} \left\{ e^{-5t} \cos(4t) \right\} = \frac{s + 5}{(s + 5)^2 + 16}$$

- Showcase the helpful little tricks we need for the inverse Laplace transforms. (Also possible as group work.)

$$1. \text{ Find the inverse Laplace transform of } F(s) = \frac{19}{s^2 + 8s + 41}$$

$$2. \text{ Find the inverse Laplace transform of } F(s) = \frac{s}{s^2 + 10s + 34}$$

- Solve some differential equations or systems. Can pick from the group work.
- Explain the steady state as the long-term behavior of a system.
- Show how to compute Laplace transforms and inverse Laplace transforms with a CAS.

**Group Work/Examples.**

- Solve a simple equation
  1.  $y'' + 4y' + 4y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$  (critically damped)  
Solution.  $y(t) = 2te^{-2t} + e^{-2t}$ .
  2.  $y'' + 8y' + 15y = e^{-5t}$ ,  $y(0) = 0$ ,  $y'(0) = 0$  (inhomogeneity is equal to one of the homogeneous solutions)  
Solution.  $y(t) = -\frac{1}{2}te^{-5t} - \frac{1}{4}e^{-5t} + \frac{1}{4}e^{-3t}$
  3.  $y'' + 4y' + 8y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$  (underdamped)  
Solution.  $y(t) = e^{-2t} \cos(2t) + e^{-2t} \sin(2t)$
  4.  $y'' + 4y' + 5y = e^{-3t}$ ,  $y(0) = 0$ ,  $y'(0) = 0$  (underdamped)  
Solution.  $y(t) = -\frac{1}{2}e^{-2t} \cos(t) + \frac{1}{2}e^{-2t} \sin(t) + \frac{1}{2}e^{-3t}$
  5.  $y'' + 4y' + 8y = \cos(5t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$  (underdamped)  
Solution.  
$$y(t) = \frac{1}{40}e^{-2t} \cos(2t) - \frac{3}{40}e^{-2t} \sin(2t) - \frac{1}{40} \cos(4t) + \frac{1}{20} \sin(4t)$$
  
(Could use this one to talk about steady state.)

6.  $y'' + 2y' + y = \cos(t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$  (critically damped)

Solution.  $y(t) = -\frac{1}{2}te^{-t} + \frac{1}{2}\sin(t)$

(Could use this one to talk about steady state.)

- Steady state solution as the long-term behavior of a real system. CAS recommended for both problems below. Common problem in application problems: Realistic numbers make the solutions quite ugly from a “classroom point of view”. But we are teaching people who will work in applications, so it helps to give a reminder that the beauty is *that* we can find a solution.

1. LRC circuit with  $L = 10mH = \frac{1}{100}H$ ,  $C = 1mF = \frac{1}{1000}F$ ,  $R = 6\Omega$  (integer frequency factor)  $E(t) = 1$ , initial conditions  $q(0) = 0$ ,  $i(0) = 0$  (gives fairly nice decomposition, can be done by hand, though the numbers are big). This one models the switching on of a circuit. Can talk about the overshoot at the beginning. Relate the overshoot to how a processor clock (rectangular frequency) can be distorted by the circuit it acts on and overshoot in places.

Solution.  $q(t) = \frac{1}{1000} - \frac{1}{1000}e^{-300t} \cos(100t) - \frac{3}{1000}e^{-300t} \sin(100t)$ .

2. LRC circuit with  $L = 25mH = \frac{1}{40}H$ ,  $C = 10\mu F = \frac{1}{100000}F$ ,  $R = 60\Omega$  (integer frequency factor)  $E(t) = \cos(2000t)$ , initial conditions  $q(0) = 0$ ,  $i(0) = 0$  (gives fairly nice decomposition, CAS recommended, though)

Solution.  $q(t) = -\frac{1}{96000}e^{-1200t} \sin(1600t) + \frac{1}{120000} \sin(2000t)$

3. Problem 46 (CAS definitely needed)

### Notable Homework Problems.

- Problems 45, 46 and 47 can be used to show how a capacitor blocks DC and conducts AC (the higher the frequency, the lower the resistance), while an inductor blocks AC (the higher the frequency, the higher the resistance) and conducts DC. (The numbers get ugly. The problems should be solved with a CAS.)

**Reading Quiz.**

1. The steady state part of a function is
  - (a) The first term of the solution.
  - (b) The part that does not converge to zero at  $t \rightarrow \infty$ .
  - (c) The constant term of the solution.
  - (d) The limit of the solution as  $t \rightarrow \infty$ .
2. Why do we need to consider Laplace transforms of  $t^n f(t)$  and  $e^{at} f(t)$ ?
  - (a) Because we know that functions of this form occur as solutions of linear constant coefficient differential equations.
  - (b) Because multiplying with an exponential will always give us another solution.
  - (c) We need powers of  $t$  to consider overdamped oscillations.
  - (d) We don't need these functions.
3.  $\mathcal{L}\{tf(t)\} =$ 
  - (a)  $-F'(s)$
  - (b)  $\int_0^s F(x)dx$
  - (c)  $sF(s) - f(0)$
  - (d)  $tF(s)$
4.  $\mathcal{L}\{e^{at} f(t)\} =$ 
  - (a)  $sF(s) - f(0)$
  - (b)  $F(s - a)$
  - (c)  $aF(s)$
  - (d)  $e^{at} F(s)$

### Section 6.5: Discontinuous Forcing Terms

**Suggested Time.** 1 class period.

**Lecture/Presentation.**

- Laplace transforms allow us access to discontinuous forcing terms in differential equations. This is something our previous methods would not allow. At the same time, the formal definition of a “solution” has to be re-evaluated and relaxed a little.
- This relaxation of the definition of a solution is natural. Mathematics was invented to model applications. So we need to shape mathematics in order to fit the applications.
- Use of the unit step function to model switching on/off phenomena.  
Model some switching on/off phenomena via step functions.
- Use of the Dirac Delta function to model (near) instantaneous energy transfer.
- The step function as the “derivative” of the Delta “function”.
- Emphasize that we keep exponentials in the Laplace transform outside the fractions to not clutter the picture.
- Solve a differential equation with initial value problem. (Also as group work.)

$$- y'' + 2y' + 2y = \begin{cases} \sin(t); & t \geq 2\pi, \\ 0; & \text{otherwise,} \end{cases} \quad y(0) = 1, y'(0) = 0$$

Solution.

$$y(t) = e^{-t} \cos(t) + e^{-t} \sin(t) + \mathcal{U}(t - 2\pi) \left[ \frac{2}{5} e^{-t+2\pi} \cos(t) + \frac{1}{5} e^{-t+2\pi} \sin(t) - \frac{2}{5} \cos(t) + \frac{1}{5} \sin(t) \right]$$

$$- y'' + 4y = 3\delta(t - 5), y(0) = 0, y'(0) = 1$$

$$\text{Solution. } y(t) = \frac{1}{2} \sin(2t) + \frac{3}{2} \sin(2t - 10)\mathcal{U}(t - 5)$$

**Group Work/Examples.**

- Why are we interested in discontinuous phenomena?
- What should the density function of a point mass be?
- Find real life situations in which an outside force is activated suddenly.
- Find real life situations in which energy is transferred nearly instantaneously.
- Why can the unit step function be considered to be an antiderivative of the Dirac Delta function?
- Write  $f(t) = \begin{cases} t^2; & \text{for } 1 \leq t \leq 2, \\ 0; & \text{otherwise,} \end{cases}$  using unit step functions.
- Find the Laplace transform of  $f(t) = \begin{cases} \sin(t); & \text{for } t \geq 4, \\ 0; & \text{otherwise.} \end{cases}$
- Find the Laplace transform of  $f(t) = \begin{cases} e^{-3t}; & \text{for } t \geq 2, \\ 0; & \text{otherwise.} \end{cases}$
- Find the inverse Laplace transform of  $\frac{e^{-3s}}{s^2 - 1}$  (re-emphasize that the exponentials are best left in front of the fractions for the inverse transform).

- Solve the initial value problem.

1.  $y'' + 6y' + 10y = 2\delta(t - 1)$ ,  $y(0) = 3$ ,  $y'(0) = 0$

Solution.  $y(t) = 3e^{-3t} \cos(t) + 9e^{-3t} \sin(t) + 2\mathcal{U}(t - 1)e^{-3(t-1)} \sin(t - 1)$

2.  $y'' + 3y' + 2y = \begin{cases} 0; & \text{for } t < 1, \\ e^t; & \text{for } t \geq 1, \end{cases}$   $y(0) = 0$ ,  $y'(0) = 1$

Solution.

$$y(t) = -e^{-2t} + e^{-t} + \frac{1}{3}\mathcal{U}(t - 1)e^{3-2t} - \frac{1}{2}\mathcal{U}(t - 1)e^{2-t} + \frac{1}{6}\mathcal{U}(t - 1)e^t$$

3. An LRC circuit has a resistor of  $80\Omega$ , an inductor of  $100\text{mH}$  and a capacitor of  $40\mu\text{F} = \frac{1}{25,000}\text{F}$ . At time  $t = 0$  we have  $i = 1\text{A}$ ,  $q = 0\text{C}$ . At time  $t = \frac{1}{100}\text{s}$

lightning strikes, causing during a negligible amount of time the transfer of  $\int E(t)dt = 4\text{Vs}$ . Find the function  $I(t)$ .

(Completing the square is easy here, though the numbers are big. Using a CAS is an option.)

Solution.  $\frac{1}{300}e^{-400t} \sin(300t) + \frac{2}{15}\mathcal{U}\left(t - \frac{1}{100}\right)e^{-400t+4} \sin(300t - 3)$

### Notable Homework Problems.

- Problem 28 gives a brief insight into what chaos theory investigates. The unperturbed system exhibits exponential decay, while the perturbed system exhibits exponential growth.

### Section 6.6: Convolutions

**Suggested Time.** 1 class period.

**Lecture/Presentation.**

- Motivate convolution via the desire to find the inverse Laplace transform of a product,
- Derive Theorem 6.61 just as presented in the text to show why the convolution is defined in this strange way
- Convolutions are used whenever the inhomogeneity of the differential equation is too hard to transform. In a way, the convolution is a last resort or a way to avoid intermediate computations.
- The price we pay in convolutions is that we will still need to solve an integral
- Side product (if you will): We can now also transform integrals, which helps when working with circuits.
- Find  $\mathcal{L} \left\{ \int_0^t \tau \cos(t - \tau) d\tau \right\}$
- Find  $\mathcal{L}^{-1} \left\{ \frac{1}{s(s - 3)} \right\}$
- Find  $\mathcal{L}^{-1} \left( \frac{s}{(s^2 - 4)(s^2 + 3)} \right)$ , if a hyperbolic function in the convolution is not desirable, change the minus in the denominator to a plus. Verify with CAS that we get the right solution.

- Can also motivate convolutions with the problem of having a rectangular wave input into an LRC circuit.

The wave will be  $f(t) = \begin{cases} 5; & \text{for } 2n \leq t < 2n + 1, \\ 0; & \text{for } 2n + 1 \leq t < 2n + 2. \end{cases}$  (Something like a processor clock.)

Our first situation shall be that  $L = 1H$  and  $C = 1F$  (no resistance).

Solution.  $y(t) = f(t) * \sin(t)$ , implemented in `processor_clock.mcd`

We see that the original signal is destroyed.

- Reality check: What kind of fool would run a processor clock through a capacitor that is the size of a small refrigerator?
- Consider the same clock sped up by a factor 1000 (call it  $g$ ) hooked up to a circuit with inductance  $1\mu H$ , resistance  $6m\Omega$  and capacitance  $\frac{10^6}{10^{12} + 9 \cdot 10^6} F$ . (Could briefly discuss that classroom problems typically use unrealistic numbers to make the solution accessible. That does not mean the world is a classroom problem. In fact, it's quite the opposite.)  
Get  $Q = G(s) \frac{10^6}{(s + 3 \cdot 10^3)^2 + 10^{12}}$ , so  $q(t) = g(t) * e^{-3000t} \sin(10^6 t)$ , which is quite small (since it is the charge). Voltage on the capacitor is  $q/C$ , so multiplication by  $10^6 + 9 \approx 10^6$  gives the voltage on the capacitor.

- Mention circuits such as AC/DC power converters and chips that input rectangular waves as the motherboard frequency. Could use the homework problems.
- Show the formula for the Laplace transform of a periodic function. Possibly solve one of the above initial value problems with it.
- Solve the integral equation  $f'(t) - 4 \int_0^t f(z) dz = 1$  with  $f(0) = 0$ .



**Group Work/Examples.**

- Project 6.7.2 “Laplace Transforms and Convolutions” can be assigned after this class.
- Solve the integral equation  $f'(t) + 4f(t) + 10 \int_0^t f(z)dz = \cos(\sqrt{10}t)$  with  $f(0) = 0$ .  
(This is the differential equation of Example 6.38 turned into an integral equation. It should be instructive to highlight the parallels after the problem has been solved. There will be differences, because this is the equation for  $i$ , not for  $q$ . That's why we have an extra  $s$  in the numerator of the Laplace transform.)
- Compute the Laplace transform of a periodic function.

**Notable Homework Problems.**

- Problem 29 shows how AC voltage is turned to DC in an AC adapter.

**Section 6.7: Projects.**

This section contains some remarks regarding surprises built into the projects.

- Project 6.7.1.
  - The equation  $2y''' + 15y'' + 30y' + 56y = \cos(t)$  gives a CAS such as MathCAD 2001i insurmountable problems, because it cannot factor the polynomial.
- Project 6.7.3.
  - Task 5c is an unsolvable initial value problem. The second equation can never be satisfied at  $t = 0$ .
- Project 6.7.5.
  - If there is no desire for a deeper excursion into circuits, parts 1 and 2 could be made optional.

**Scoring Rubric/Checklist for Laplace Transform Projects in Which a Program Automatically Solves Equations**

The scoring rubric below is approximately what will be used to score the Laplace transform projects. It can be used as a check sheet to determine completeness of the project. While completeness according to this rubric does not guarantee a perfect score, incompleteness certainly guarantees an imperfect score.

“Live” computation means that, once the problem is entered, all other steps are computed without adjustment by the programmer.

To be handed in for each project:

- One printout of the program (as inputs, use the values of task 1),
- Printed solutions for all tasks (not the programs themselves),
- For tasks that could not be completed, a written explanation why the program did not work.
- A disk containing for each task the program with the appropriate input.

Assessment item	percentage	check/score
Inputs encoded correctly	10%	
Laplace transforms of right side(s) computed “live”	10%	
Laplace transforms of solution(s) correct (supply preparatory hand computation)	15%	
Partial fraction decomposition of solution(s) computed “live”	5%	
Actual solution(s) computed “live”	5%	
Graph of solution(s)	10%	
Check if solution(s) solve the differential equation	10%	
Check if solution(s) have the right initial value	10%	
Correct solutions to the tasks	10%	
Fixed problems with program in tasks where the program got stuck (if applicable)	10%	
Grammar/Style/Effective presentation of computations	5%	

**Before Module 7: Examples of PDEs, Initial and Boundary Conditions.****Suggested Time.** 0.5-1 class.**Lecture/Presentation.**

Some example of a PDEs should be presented, with derivation or at least motivation. It would be counterproductive to attempt presenting all derivations in the text. Rather than that, I would recommend to the students to determine which examples are closest to their fields and to read these independently.

- Partial differential equations are equations involving partial derivatives of functions. Just like for ordinary differential equations, a solution is a function that, when substituted into the equation, makes the equation true.
- Partial differential equations can be used to model phenomena that depend on time *and* space.
- Show that  $u(x, y) = \cos(x^2 + y^2)$  solves the PDE

$$u^2 + \left( \frac{\partial u}{\partial x} \right)^2 = 1$$

(for  $x \neq 0$ )

- Show how the equation of a vibrating string comes about.
- Show how the heat equation comes about. (Also consider the activity on page 331. Careful with the first integral. A positive integral would be an outflow, so we need another negative sign there.) Note that this equation governs other types of transport phenomena.

The following is a generic, brief (and quantitatively incomplete) way to explain the heat equation. If the right ideas from multivariable calculus are settled in students' minds, it may be a good counterpoint to the derivation in the text: If  $u$  is temperature, then  $-\nabla u$  is heat flux and  $\Delta u$  is the source strength of the heat flux. The only way to generate a heat flux without outside sources is through local temperature changes.

Thus  $\Delta u = k \frac{\partial}{\partial t} u$ , where  $k$  is a positive constant depending on the material. The constant  $k$  is positive, because cooling down represents an outward flow of energy (which means without sources, the gradient points inwards and the divergence is negative; the trip-up here is that the gradient points in the direction of steepest ascent, which is *against* the flow of energy), while heating up represents an inward flow of energy (similar reasoning).

This generic explanation also allows to adapt this idea to diffusion equations where temperature is replaced by concentration of a certain substance.

- Mention the Schrödinger equation with brief explanation of what a wave function is. This can only be the briefest of starts, giving students the overall idea that quantum mechanical phenomena are strange, but real.
- In the abundance of solutions for PDEs, how do we know which solutions are useful? (Strange solutions are for example  $e^{x+y+z+3t}$  solving the heat equation,  $x^2 + t^2$  solving the one-dimensional wave equation, etc.)

What plays the role of the initial conditions in ODEs?

- Notice that PDEs *must* combine IVPs and BVPs since they often deal with space and time.
- Discuss the initial and boundary conditions for the vibrating string.

- Show that  $u(x, t) = \sin(10x) \sin(10t)$  satisfies the initial condition  $u(x, 0) = 0$ ,  $\frac{\partial}{\partial t}u(x, 0) = \sin(10x)$ , the boundary condition and the PDE

$$\frac{\partial^2}{\partial x^2}u = \frac{\partial^2}{\partial t^2}u.$$

### Group Work/Examples.

- Show that  $u(x, y) = x^2 \sin(y)$  solves

$$\frac{1}{2}x^2 \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 0$$

- Interpretation of solutions of the heat/wave/other “real-life” equation. Fix  $t$  or the space parameter and explain what the thus obtained function describes.
- Activity on page 331.
- Does  $u(x, t) = x(x - \pi) \cos(t)$  satisfy the initial condition  $u(x, 0) = x(x - \pi)$ ,  $\frac{\partial}{\partial t}u(x, 0) = 0$ , the boundary condition and the PDE

$$\frac{\partial^2}{\partial x^2}u = \frac{\partial^2}{\partial t^2}u?$$

- What are the initial and boundary conditions for a ball that is heated through its surface (egg in boiling water)?

### Reading Quiz.

- The purpose of using partial differential equations is
  - Description of static, higher-dimensional phenomena,
  - Description of one-dimensional, time-dependent phenomena,
  - Description of higher-dimensional, time-dependent phenomena,
  - All of the above.
- The Laplace operator can be interpreted as
  - An operator that produces the sources of a vector field.
  - An operator that measures temperature.
  - An operator that measures the electromagnetic field.
  - An operator that produces the source strength of the flux of a scalar function.
- Identify the three dimensional heat equation among the following equations.

(a)  $\Delta u \cdot k \frac{\partial}{\partial t}u = 0$

(b)  $\Delta u \cdot k \frac{\partial^2}{\partial t^2}u = 0$

(c)  $\Delta u - k \frac{\partial}{\partial t}u = 0$

(d)  $\Delta u - k \frac{\partial^2}{\partial t^2}u = 0$

### Section 7.1: A Simple Start: Separation of Variables

**Suggested Time.** 1 class.

**Lecture/Presentation.**

- Solve a cooked up first order PDE with IVP and BVP to exhibit the method.

$$x \frac{\partial}{\partial x} u(x, y) - 2 \frac{\partial}{\partial y} u(x, y) = 0$$

with  $u(x, 0) = x^2$ .

(sol.  $u(x, y) = x^2 e^y$ )

- Note that while the variables are separated, they still “communicate through  $\lambda$ ”
- Refer to Section 3.7 for the boundary value problem  $y'' + \lambda y = 0$ ,  $y(0) = y(b) = 0$  that comes up in one-dimensional heat and wave equations.

**Group Work/Examples.**

- Solve the PDE with given boundary condition.

$$1. \frac{t}{t^2 + 1} u_x - u_t = 0, u(x, 0) = 3e^{2x}.$$

$$2. \frac{u_x}{1 - 2x} + \frac{u_t}{x - x^2} = 0, u\left(\frac{1}{2}, t\right) = e^{-t}$$

$$\text{Solution: } u(x, t) = 4x(1 - x)e^{-t}$$

$$3. \frac{u_x}{x(y^2 - 1)} - \frac{u_y}{2y(x^2 + 1)} = 0, u(1, 0) = 2, u(2, 0) = 5$$

$$\text{Solution: } u(x, y) = (x^2 + 1)(y^2 - 1)^2$$

- Solve a one-dimensional heat equation with IVP and BVP. Use conditions similar to what is in the text for the vibrating string. (Possible lead-in to discussing the project.)

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$$

On  $[0, \pi]$ .  $u(x, 0) = \sin(x)$ ,  $u(0, t) = u(\pi, t) = 0$

**Notable Homework Problems.**

- Problem 15 shows that the solution to the wave equation is not always the product of the initial condition with a trigonometric function. This counteracts a possible impression given by earlier problems that work out nicely.

### Section 7.2: Fourier Series

**Suggested Time.** 1 class period

#### Lecture/Presentation.

- Fourier series are motivated by attempting to solve PDEs with separation of variables.
- Fourier series representations only approximate on a bounded interval and then periodically extend the function.
- Present Theorem 7.7 with CAS. (Can also make students do this in groups.)
- Prove Theorem 7.8.
- Use the Fourier coefficients of  $f(x) = x^2$  to show that the resulting function is now  $2\pi$ -periodic.
- Show that the second derivative of the Fourier series of  $f(x) = x^2$  does not approximate the actual second derivative (oscillations!).

#### Group Work/Examples.

- Compute the Fourier coefficients of a function like  $f(x) = |x|$  or  $f(x) = x^2$ .
- Plot the Fourier polynomials of the function for which Fourier coefficients were computed. Note that the resulting graph is periodic, independent of whether the function is or not.
- Why must functions that we want to represent with Fourier series be  $2\pi$  periodic?

#### Reading Quiz.

- In Fourier series, functions are represented as
  - Sums of powers of  $x$ ,
  - Sums of trigonometric functions,
  - Sums of real exponential functions,
  - None of the above.
- $\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx =$ 
  - 0 for all  $n$  and  $m$ ,
  - $\pi$  for all  $n$  and  $m$ ,
  - 0 for  $n = m$ ,  $\pi$  for  $n \neq m$ ,
  - None of the above.
- True or False (T/F).
  - If  $f$  is a differentiable  $2\pi$ -periodic function, then  $f$  is the limit of its Fourier series for all  $x$ .
  - Fourier series are used to solve initial value problems for partial differential equations.
  - Fourier series are used in computers to compute the values of functions such as  $e^x$ .

**Section 7.3: Fourier Series and Separation of Variables**

**Suggested Time.** 0.5-1 class.

**Lecture/Presentation.**

- Straight presentation of the section.

**Group Work/Examples.**

- Some of the integrals for the Fourier coefficients could be solved in groups.

**Section 7.4: Bessel and Legendre Equations**

**Suggested Time.** 0.5-1 class.

**Lecture/Presentation.**

- Straight presentation of separation of variables for Bessel equations, Legendre equations or both.



### Section 8.1: Expansions About Ordinary Points

**Suggested Time.** 1 – 2 class periods.

**Lecture/Presentation.**

- Even a differential equation as simple as  $y'' - xy = 0$  (an Airy equation) has no solution in closed form (whatever “closed form” may mean). Yet from the theory of linear differential equations we know there must be many solutions.
- Series solutions allow us to solve differential equations that are not accessible through setting up  $e^{\lambda t}$  or via Laplace transforms.
- The differential equations that we solve with series normally come out of partial differential equations via separation of variables. (They can also arise directly, but partial differential equations are the overwhelming source.)
- Solve  $y'' + 3xy' + xy = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$  using series. Show how to check the solution.

$$y(x) \approx 1 - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{180}x^6 - \frac{3}{112}x^7 - \frac{1}{320}x^8 + \dots$$

- Even if we only have the first few terms of the solution, we can check if we have a good approximation. Plugging in the first few terms of the expansion should lead to an error term of high order in the independent variable.
- Define analytic functions.
- Define singular points.
- Present Theorem 8.10.

**Group Work/Examples.**

- Activity on page 325.
- Solve the differential equation (or initial value problem) using series. Write the first few terms of the series solution. (Five nonzero terms seems to be a pretty good number.) Check if the solutions are correct by either plugging the series into the differential equation (if there is a pattern) or by using the first few terms (if there is not). In case we look for general solutions, give two independent solutions. (One with  $c_0 = 1$ ,  $c_1 = 0$ ; the other with  $c_0 = 0$ ,  $c_1 = 1$ .)

1.  $y'' - xy = 0$ , (recurrence relation:  $c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}$ )

2.  $y'' - y' + xy = 0$ , (recurrence relation:  $c_{n+2} = \frac{(n+1)c_{n+1} - c_{n-1}}{(n+2)(n+1)}$ )

3.  $(x-2)y'' + y' - y = 0$ ,  $y(0) = -3$ ,  $y'(0) = 2$

$$y(x) \approx -3 + 2x + \frac{5}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{24}x^4 + \frac{1}{96}x^5 + \frac{7}{1920}x^6 + \frac{29}{20160}x^7 + \frac{11}{18432}x^8 + \dots$$

- Find the singular points of

1.  $y'' + \frac{y}{x} = 0$

2.  $e^x y'' + (x^2 + 1)y' + \cos(x)y = 0$

3.  $xy'' + y' + x^2y = 0$

4.  $xy'' + \sin(x)y = 0$

5.  $(x^2 - 1)y'' + (x - 1)^2y' + (x - 1)y = 0$

6.  $(x^2 + 1)y'' + y' + (x - 1)y = 0$

- Find the radius of convergence of a power series solution for the differential equation  $(x^2 + 1)y'' + xy' - y = 0$ 
  1. Expanded about 0
  2. Expanded about 1,
  3. Expanded about 5.
  
- Find the singular points of the equation  $(x^2 + 4)y'' + \frac{y'}{x + 2} - \frac{y}{x + 5} = 0$  and use them to give a lower bound on the radius of convergence for series solutions expanded about 0 and 5.

**Section 8.2: Legendre Polynomials and Special Functions**

**Suggested Time.** 0.5 – 1 class periods.

**Lecture/Presentation.**

- Straight presentation of series solution for Legendre equations. (Can be presented as another example to illustrate series solutions.)
- If Module 7 was not covered, a quick intro to the Legendre equation via separation of variables would be appropriate.
- Emphasize the need for pattern recognition skills. But also note that for many equations there is no pattern or the pattern is too complicated.

### Section 8.3: Expansions about Singular Points.

**Suggested Time.** 1 – 2 class periods.

#### Lecture/Presentation.

- It is possible to expand about singular points, though the method has to be adjusted.
- Recall that the Bessel equation arises in solving heat and wave equations with circular domains. Note that 0 is a singular point.
- Define (regular) singular points. Motivate the idea with a Cauchy-Euler equation, such as  $x^2y'' + xy' - 4y = 0$ .
- Present Frobenius' theorem.
- Solve  $2x^2y'' - xy' + (x + 1)y = 0$  about  $x_0 = 0$ .

$$c_k = \frac{-c_{k-1}}{2(k+r)(k+r-1) - (k+r) + 1} \text{ One solution with } r = 1 \text{ and one solution with } r = \frac{1}{2}.$$

$$y_1(x) = x - \frac{1}{3}x^2 + \frac{1}{30}x^3 - \frac{1}{630}x^4 + \frac{1}{22680}x^5 - \frac{1}{1247400}x^6 + \dots,$$

$$y_2(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}} + \frac{1}{6}x^{\frac{5}{2}} - \frac{1}{90}x^{\frac{7}{2}} + \frac{1}{2520}x^{\frac{9}{2}} + \dots$$

- Solve (parametric) Bessel Equations or a special case thereof.

#### Group Work/Examples.

- Activity on page 327.
- Find and classify the singular points.
  1.  $x^2y'' + 3xy' + y = 0$
  2.  $x^2y'' + 3y' + 4x^2y = 0$
  3.  $(x^2 - 4)y'' + 3(x - 2)y' + 4(x - 2)^2x^2y = 0$
- Solve the differential equation by expanding about the singular point  $x = 0$ .
  1.  $x^2y'' - (x + x^2)y' - 3y = 0$ ,  $r = -1, 3$ ,  $c_n = \frac{c_{n-1}(n+r-1)}{(n+r)((n+r)-2) - 3}$ ,  $r = -1$  leads to the solution  $y_1(x) = \frac{1}{x} + \frac{1}{3}$ ,  $r = 3$  gives the other solution.
 
$$y_2(x) \approx x^3 + \frac{3}{5}x^4 + \frac{1}{5}x^5 + \frac{1}{21}x^6 + \frac{1}{112}x^7 + \frac{1}{720}x^8 + \dots$$
  2.  $xy'' + 3y' - y = 0$ ,  $r = 0, -2$ ,  $c_{n+1} = \frac{c_n}{(n+1)(n+3)}$ ,  $r = -2$  leads to a division by zero.
 
$$y_1(x) \approx 1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \frac{1}{8640}x^4 + \frac{1}{320400}x^5 + \dots$$
  3.  $x^2y'' + 4xy' + 2y = 0$  (sol.  $x^{-1}$  and  $x^{-2}$ , Cauchy-Euler equation)
  4.  $x^2y'' + xy' + (x^2 - 2)y = 0$  (Bessel equation with  $\lambda = 1$  and  $\nu = \sqrt{2}$ )
  5.  $x^2y'' + (x^2 + x)y' - y = 0$ ,  $r = \pm 1$ 

$$y_1(x) \approx x - \frac{1}{3}x^2 + \frac{1}{12}x^3 - \frac{1}{60}x^4 + \frac{1}{360}x^5 - \frac{1}{2520}x^6$$
  6.  $x^2y'' + (x - \frac{1}{3})y' - \frac{1}{3}y = 0$

**Section 8.4: Bessel Functions**

**Suggested Time.** 0.5-1 class.

**Lecture/Presentation.**

- Straight presentation of Frobenius solution for Bessel equations. (Can be presented as another example to illustrate Frobenius method.)

**Section 8.5: Reduction of Order.**

**Suggested Time.**  $\frac{1}{2}$  – 1 class period.

**Lecture/Presentation.**

- If a method such as Frobenius' method does not give all solutions to a differential equation, one can under certain circumstances still obtain the second solution if one has at least one solution.
- Possible lead in with  $xy'' + 3y' - y = 0$ ,  $r = 0, -2$ , for  $r = 0$ :  $c_{n+1} = \frac{c_n}{(n+1)(n+3)}$ , however,  $r = -2$  leads to  $c_{n+1} = \frac{c_n}{n^2 - 1}$  and  $y(x) = x^{-2} - x^{-1}$  if we hope that the only two coefficients we can compute really are good enough to get a solution. This hope is easily debunked.

Note: The differential equation here is also investigated in Project 8.6.3 and there is therefore some overlap.

- Present the reduction of order formula.

**Group Work/Examples.**

- Apply the reduction of order formula/method to find another solution.
  1.  $6y'' + y' - y = 0$ ,  $y_1 = e^{\frac{x}{3}}$ ,
  2.  $x^2y'' - 7xy' + 16y = 0$ ,  $y_1 = x^4$ .
  3.  $x^2y'' + xy' - 4y = 0$ ,  $y_1 = x^2$ .
- Use a computer algebra system and compute more than just the first three coefficients for Example 8.33.

**Section 9.1: Existence and Uniqueness of Solutions**

**Suggested Time.**  $\frac{1}{2}$  class period.

**Lecture/Presentation.**

- Define systems and their solutions.
- Explain that all systems can be translated to first order systems (Theorem 9.2).
- Introduce the matrix notation for systems as an abbreviation that is similar to the abbreviation of systems of (algebraic) linear equations using matrix notation.

**Group Work/Examples.**

- Show that  $y_1' = 2y_1$ ,  $y_2' = y_1 + y_2$  is solved by the functions  $y_1(t) = e^{2t}$  and  $y_2(t) = e^{2t}$ .
- Translate the differential equation  $y'' + 3y' + 2y = 0$  into a first order system of differential equations.  
Then translate the solutions of the equation to solutions of the system.
- Translate the systems above into matrix notation.
- Discuss existence and uniqueness of solutions, fundamental sets and Wronskians.

**Section 9.2: Matrix Algebra**

**Suggested Time.**  $\frac{1}{2}$  class period.

**Lecture/Presentation.**

- Explain matrix multiplication.
- Introduce the identity matrix and inverses. Emphasize that we want simple-looking algebra that encodes the rather complicated operations.
- Show how these operations can be performed by a CAS.

**Group Work/Examples.**

- Multiply the matrices  $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 4 & 0 \\ 2 & 3 & -1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & 4 \end{pmatrix}$  in both possible ways.
- Compute the inverse of the matrix  $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}$ .



### Section 9.3: First Order Systems with Constant Coefficients – Diagonalizable Case

**Suggested Time.** 1 class period.

**Lecture/Presentation.**

- Explain why systems with a diagonal coefficient matrix are easy to solve.
- Explain Theorem 9.24 to show that with the right transformation certain systems of equations will become easy to solve.
- Define eigenvectors as the vectors that give us the transformation matrix (they are the columns).
- Explain how we can find eigenvalues as the zeroes of the characteristic polynomial.
- Explain what to do for complex eigenvalues.
- Explain what to do for repeated eigenvalues.
- Explain how to solve initial value problems.

**Group Work/Examples.**

Note that the  $3 \times 3$  examples take significantly longer than the  $2 \times 2$  examples. could incorporate an initial value problem into each of the examples below.

- Activity on page 329.
- Solve the system of differential equations  $y' = \begin{pmatrix} -2 & 4 \\ -6 & 8 \end{pmatrix} y$ . (Distinct eigenvalues.)
- Solve the system of differential equations  $y' = \begin{pmatrix} 3 & 2 & -2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix} y$ . (Distinct eigenvalues.)
- Solve the system of differential equations  $y' = \begin{pmatrix} 1 & 1 & -3 \\ -3 & 5 & -3 \\ -3 & 3 & -1 \end{pmatrix} y$ . (Distinct eigenvalues.)
- Solve the system of differential equations  $y' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} y$ . (Complex eigenvalues.)
- Solve the system of differential equations  $y' = \begin{pmatrix} 3 & 3 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix} y$ . (Complex eigenvalues.)
- Solve the system of differential equations  $y' = \begin{pmatrix} 1 & 3 & -3 \\ 3 & 1 & -3 \\ 3 & 3 & -5 \end{pmatrix} y$ . (One double eigenvalue.)
- Solve the system of differential equations  $y' = \begin{pmatrix} -1 & 1 & -3 \\ -3 & 4 & -3 \\ -3 & 3 & -1 \end{pmatrix} y$ . (One double eigenvalue.)

**Section 9.4: Non-Diagonalizable First Order Systems with Constant Coefficients**

**Suggested Time.** 0.5 – 1 class periods.

**Lecture/Presentation.**

- Present Theorem 9.37.

**Group Work/Examples.**

- Solve the system of differential equations  $y' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} y$ .
- Solve the system of differential equations  $y' = \begin{pmatrix} -4 & -2 & 8 \\ 1 & 2 & -1 \\ -3 & -1 & 6 \end{pmatrix} y$ .

**Section 9.5: Qualitative Analysis**

**Suggested Time.** 0.5 class period.

**Lecture/Presentation.**

- Quick overview of how the eigenvalues influence the shape of the trajectory.

**Group Work/Examples.**

- Discuss the qualitative behavior of the solutions of any prior classroom example or homework problem. Then plot the trajectory of the analytical solution to double check.

**Section 9.6: Variation of Parameters**

**Suggested Time.** 0.5 – 1 class periods.

**Lecture/Presentation.**

- Present the Variation of Parameters formula.
- Use the formula for systems to derive the formula for second order equations.

**Group Work/Examples.**

- Find the general solution of the system

$$\vec{y}' = \begin{pmatrix} -11 & -24 \\ 4 & 9 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ t \end{pmatrix}$$

**Section 9.7: Outlook on the Theory: Matrix Exponentials and the Jordan Normal Form**

**Suggested Time.** 0.5 class period.

**Lecture/Presentation.**

- Quick overview of how differential equations lead to some natural questions in linear algebra and how the Jordan Normal Form is part of the answer.