Appendix MAN

Instructor's Manual

This instructor's manual grew out of class preparations that I started typing when I began my leadership role in Louisiana Tech University's integrated curricula. My handwriting is atrocious and I always lose notes on paper. On a computer I can edit, I type fast and files do not get lost. Moreover, after piloting new classes, I thought colleagues might benefit from a set of suggestions. Nothing in this manual should be considered a prescription and suggestions for additions/modifications are appreciated. Colleagues' use of the preparations pointed out how a manual such as this one can be effective. They used examples they liked, sometimes my order of presentation with their examples, sometimes (emergency situation, little preparation time) they used the whole preparation and sometimes they used none of it. Because the manual has grown as I added examples after new ideas arose, I should also say that I don't always cover every word in the manual.

For every section of the text, the following information is provided.

Suggested Time. This is an obvious starting comment. In any text, some sections will need more class time, others less.

Lecture/Presentation. This section briefly states the main points. In some cases the list may not be exhaustive, because certain lecture elements can be considered obvious.[1] Moreover, some points may just be key words ("present or prove" a certain theorem). As noted above, the order may not be the order of presentation. Moreover, to encourage more classroom interaction, I have moved most of the examples into the "group work" section. So when using a "lecture" section that has no examples for presentation, I usually pull a few examples out of the "group work" section or also make up examples on the spot.

I use on-line demonstrations to quickly make a point that I cannot make with a marker and a board. I show the animation or applet, make the point and get back to business. We can enhance visual and kinesthetic learners' understanding this way, but ultimately mathematics is learned by doing, not by watching videos. (It would be great if the "videos" would spontaneously materialize in the students' heads, not be played on a screen. That would take care of any problem in which the key is to "visualize it".) Some of these applications are in the "class materials - animations" section of my home page at `http://www.LaTech.edu/~schroder`. For high quality electronic resources, also consider MERLOT, the Multimedia Educational Resource for Learning and On-line Teaching at `http://www.merlot.org`. Finally, some direct links are listed in the manual.

[1] For example, it goes without mentioning that every new integration technique is followed by computational examples.

Group Work/Examples. Independent of the bad publicity lectures seem to be getting at times, lectures will remain our primary mode of presentation. I like Rich Felder's attitude in his cooperative learning workshops. He explicitly points out that talking about cooperative learning does not mean lectures are bad and he himself still lectures about 85% of the time. Cooperative learning can make lectures more effective, though.

The easiest way to loosen up a lecture is to not present every single last example. My colleague Jenna Carpenter often says (and I agree) that "students get more out of three examples they do themselves than out of ten that I do on the board." For many fundamental techniques it suffices to present one, two or no examples on the board and then turn the students loose with more examples for them to solve. All these examples can be found in this section. Of course it would not help to let students flounder too much. I usually observe students as they work and help them catch mistakes or to continue when they are stuck. I also quietly write solutions on the board for problems that a majority seems to have solved. Periodically I talk them through the solution, putting emphasis on difficulties that I observed and how to circumvent these difficulties. Sometimes the observation also leads me to curtail or interrupt the group work if something just isn't working out. For more suggestions how to structure group work (if so desired), please consider the introduction to Module ACT (separate downloadable document).

In some cases, a class that is focused on one topic, say, integration by substitution, can be largely turned over to the students. Indeed, using the "Discovery and Practice" activity for substitution on page ACT-29, I say very little in class. They do the discovery part, I summarize, present theory and an example, then they practice and I periodically talk them through a solution. Other classes will require more instructor's presentation. There is no hard and fast rule how much should be lecture and how much should be student work. If it works, lecture the whole time and present the group work examples here if you like them. Or, if it works, let students run the whole class in a Moore method style. Somewhere between these two extremes is your teaching style. A class will be good as long as the emphasis remains on making it work rather than on implementing a certain philosophy that may or may not apply. After all, every educational method will work great with one group of people and not at all with another group. The trick is to find what works best with the class presently taught, taking into account that the real life problems we aim to solve do not change because of the solver's learning style.

Reading Quiz. My colleague Lee Sawyer introduced me to the idea of a reading quiz. To get students to read the book before class, a short (at most 5 minutes) multiple choice quiz that covers the highlights of the section can be given before class starts. This quiz should be set up so that a casual reader will be able to answer the questions. Perfect preparation is not necessary. After all, what would be the point in coming to class then? Since the reading quizzes are often simple they should not carry a large amount of credit (I despise grade inflation). Some credit is appropriate to give the proper motivation, though.

Preparation for the reading quiz readies the mind for the detailed work to come in class. The students are exposed longer to the material and will be able to think more deeply about it in class, which overall should improve performance. I am still struggling with finding good reading quizzes and hence this section is often empty Yet I believe the idea is worth a shot. If you have a good set of reading quizzes for a class, I would love to see them.

Notable Homework Problems. The component of a class in which most learning happens probably is the homework. Here students must do the problems. Points where they might get stuck cannot be deferred to the instructor's authority in class.

Homework problems to me are "notable" if they are

- Not found in a typical text, or
- Highly realistic, or
- Part of an underlying strand, such as a problem that is revisited several times, that might be overlooked otherwise, or
- Requiring the student to refer back to the text and read something in detail, etc.

These problems are listed under this heading and some explanation what makes them "notable" may be given. This does not mean that the other problems are bad. It just means that these problems stand out to me. Like the reading quizzes, but to a lesser degree, the sections on notable homework problems are also a bit unfinished. As I revisit homework sections for class preparation, these "notable" sections will grow.

Section 1.1. Basic Functions and Their Graphs

Suggested Time. 1 class period.

Lecture/Presentation.

- Define a function as a "rule that assigns to each element x of a set X (called the **domain**) a *unique* element y of a set Y (called the **range**)"
  - Examples: candy machine, stock of certain candy at grocery store, fuel left in car (dep. on miles driven)
  - y: dependent variable (output) depending on
    x: independent variable (input)
    (use a "black box picture"?)
  - Independent variable often is time. For example the height of a human over time. (Draw height over a day: decreasing during the day, increasing at night, because of minute compression and decompression of disks between vertebrae.)
  - Functions are often represented by their graph, which is the collection of points $(x, f(x))$ where x is in the domain.
    Recall graph of an equation and demonstrate the vertical line test.
    Graph $f(x) = 2x^3 - 4x + 2$ and $x^2 + y^2 = 1$ on a CAS.
  - When a few precise values are needed, tables are an option. Use any table in the appendix as an example.
    Good connection to computer science. In many operating systems certain functions are implemented via lookup tables (plus interpolation). For an example, consider
    `http://physinfo.ulb.ac.be/cit_courseware/datas/data3.htm`
  - Connection to chemistry. The periodic table contains a multitude of functions. The domain are the elements (same word as in definition of function, so we can be cute), the range(s) are the various numbers (atomic number, mass, electron configuration the full name or the abbreviation).
    Can foreshadow injectivity by asking how things would change if we ordered by mass instead of atomic number (there will be ties, Curium and Berkelium's most stable isotopes have mass 247, likely just as good as we can measure.) A periodic table can be found at
    `http://www.dayah.com/periodic/`

- The multiple ways a function can be represented can be compared with the different data structures for the same type of object in computer science. Choice of the appropriate structure will make the task itself easier.

  Also for CS students it is worth mentioning that there is a programming language (LISP) that is based entirely on functions.

- Evaluate $f(x) = 2x^3 - 4x + 2$ at -1, 3.141, $x + 2$ and give a verbal description.
  - For evaluation at $x + 2$ use the open parentheses method and translation into words

- Note that the verbal description helps when the independent variable has a name other than x (extreme case $x = f(y)$) or when x's are substituted for x's (as in $f(x+2)$).

- Evaluate a function given by a graph.
- Evaluate a function given by a table.
- A CAS can be used when all we want to do (or can do) is plot many points. One has to be careful with the window, though.

  Could use $f(x) = x^5 - 12x^4 - x + 2$ in the standard window $[-10, 10] \times [-10, 10]$ to show that some graphs do not tell the whole story.

- Find the domain of $f(x) = \dfrac{1}{x^2 - 4}$. First draw the domain, then write it in set builder notation, then with intervals.

  - In set builder notation we define a set of numbers by opening a set brace followed by an x and a vertical line, followed by the condition the points are to satisfy followed by a closed set brace. (Very verbose, do it graphically: write down a set and show with arrows how to read it.)
  - Explain interval notation: An interval is denoted by two numbers separated by a comma and enclosed by parentheses.
    * The first number is the start of the interval,
    * The second number is the end of the interval,
    * A square brace says that the endpoint is included,
    * A round brace says the endpoint is excluded.

- The natural domain is the set of numbers where the function can be evaluated.
- The graphs of standard functions are patterns that occur frequently.
- Define x- and y-intercepts and show how to find them.
- Mention the importance of the difference quotient in calculus, importance of being solid in algebra.

Group Work/Examples.

- Find examples of real life functions, specify independent and dependent variable,
- For $f(x) = 2x^2 - x + 1$ find $f(2)$, $f\left(\dfrac{1}{4}\right)$, $f(9.2418)$ (calculator), $f(x + h)$, $\dfrac{f(x+h) - f(x)}{h}$;
- What is the domain of the function that assigns to each possible radius the area of a circle of that radius? What is the domain of the function $A(r) = \pi r^2$?

  For some domains we need to know the context, since the formula can also be evaluates for values that make no sense in the problem (domain of the function vs. domain of the problem)

- Write the domains of the following functions in set-builder and interval notation

  1. $f(x) = \sqrt{6x - 7}$,
  2. $g(u) = \dfrac{1}{u-1} + \dfrac{1}{u+1}$,
  3. $h(t) = \dfrac{\sqrt{3-t}}{t^2 - 3}$.

- One of the "Dr. Absentminded" problems.

- Do one of the "Catching mistakes" problems.

Reading Quiz.

Notable Homework Problems.

- Problem 9 requires students to make sense of the explanation for the normal distribution table. Since no other explanation is given, this is actually a pretty deep problem at that stage, even though it is trivial for those familiar with elementary statistics.

- Problem 12 foreshadows the development of instantaneous velocities.

Section 1.2. Shifting, Stretching and Reflecting

Suggested Time. 1 class period.

Lecture/Presentation.

- Vertical shifts: Draw $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x} + 4$, they $h(x) = \sqrt{x} - 3$,
- Horizontal shifts Draw $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x+2}$, they $h(x) = \frac{1}{x-5}$,
- Explanation for horizontal shifts: Find the function f in $f(x+c)$. Then find where $x + c = 0$. That is where the old origin goes.
- Reflection about the x-axis $f(x) = \sin(x)$ vs. $f(x) = -\sin(x)$. Even if the sine function is not formally introduced, the principle can be understood.
- Reflection about the y-axis. $f(x) = 3x + 5$ vs. $f(x) = 3 \cdot (-x) + 5$.
- Stretching and compressing in the y-direction (scaling) $f(x) = x^3 - 5x$ vs. $f(x) = 5\left(x^3 - 5x\right)$, $f(x) = \frac{1}{3}\left(x^3 - 5x\right)$, $g(x) = -3\left(x^3 - 5x\right)$.
- Up to here, all suggestions are also incorporated in the activity on p. ACT-1, so the above could be skipped in favor of the activity.
- Draw three graphs. One even, one odd, one without symmetry and ask if they can recognize anything in these graphs. If they don't see anything mention the idea of symmetry.
- Define even and odd functions as functions that are "immune" against (i.e., invariant under) certain transformations.
- Define composition $f \circ g(x) = f(g(x))$ for all x such that x is in the domain of g and $g(x)$ is in the domain of f

  Picture with output of one box becoming the input of another might also help.

  Computer science example. The original input "source code" is turned into intermediate files by the compiler, which are turned into the final executable by the linker.

  Engineering/Science example. The original input "real life data" is turned by a sensor into binary data that is sent to a computer, which can turn it into a table or a graph.

- Show composition of $f(x) = \frac{1}{\sqrt{2\pi}} e^x$ (this function is formally unknown, so it might be a good idea to present the graph and predict the composition graphically) and $g(x) = -\frac{x^2}{2}$ with domain and range check
- Domain of $f \circ g$ is the set of x for which $g(x)$ and $f(g(x))$ are defined.
- Graphical example of composition. f: Line through origin and $(1, 2)$ and g: semicircle with center at origin and radius 2. **Just use the graphs of these functions or use graphs from which we cannot "reverse engineer" the formula.**

  Find

1. $g \circ f(0)$
2. $g \circ f(2)$,
3. $f \circ g(2)$

Group Work/Examples.

- Activity on page ACT-1.

  While students do the activity, develop a table that sums up the rules.

- For a graphically given function f, find $f(x-2)+3$.

- A recognition problem as in Exercise 4 would drive home the point that we need to know the graphs of the standard functions and how they shift.

- Think-Pair-Share. Identify the function as even odd or neither. Do the functions given by a formula symbolically.

  1. Some graphs, even, odd, neither, plus some that are close, such as odd function shifted up.
  2. $f(x) = x^2 + 2$
  3. $f(x) = x^3 - 1$
  4. $f(x) = 3x + 2x^3$
  5. $f(x) = \sin(x)$ (only those students who know the trigonometric functions)
  6. Some functions given by tables

- For $f(x) = \sqrt{x-1}$ and $g(x) = x^2$ find the rules and domains of the following functions

  1. $f \circ g$
  2. $g \circ f$
  3. $f \circ f$
  4. $g \circ g$

- Write the function $f(x) = \sqrt{(x+1)^2 - 4}$ as the composition of as many functions as possible. (Find the record in the class.)

Reading Quiz.

Notable Homework Problems.

- Exercise 17d is the solution to exercise 5. It may be interesting to assign both problems in the same homework set.

Section 1.3. Modeling with Functions

Suggested Time. 1 class period.

Lecture/Presentation.

- Introductory statement "Today is a class in which we will learn no new facts, ... which is exactly why today's class is perceived as hard."

- Polya's (semi-)structured approach to problem solving. "Define, plan, execute, evaluate". Talk about the solving process and why it would be counterproductive to attempt to give too many details (you limit yourself to certain problem types as details increase). Could give an alternative set of instructions.

  1. Read the problem,
  2. Draw a picture,
  3. Name the quantities,
  4. Derive a formula,
  5. Solve the mathematical problem,
  6. Check if the answer is sensible.

- The product of one number with the square of another is 2000. Find a formula for the sum of the two numbers that depends only on the number that is squared.

  How big is the sum if the squared number is equal to 10 (so its square is 100, possibly add if there is a question which number is meant)?

  How can we make the sum bigger?

- In the problems in this section sometimes values for the independent variable are given to describe the current situation/plan. Students should not consider these numbers fixed. This situation is similar to the challenges some students face with related rates problems.

- A function is called **(strictly) increasing** in an interval if and only if for all $a < b$ in the interval we have $f(a) < f(b)$.

- A function is called **(strictly) decreasing** in an interval if and only if for all $a < b$ in the interval we have $f(a) > f(b)$.

- A function is called **constant** in an interval if and only if for all a, b in the interval we have $f(a) = f(b)$.

- Sketch a graph and ask where the function is increasing or decreasing.

- A rectangular box with square base has volume $2000in^3$. The sidelength of the base is $10in$. To decrease the surface area of the box, should the height be decreased or increased?

Group Work/Examples.

- A farmer wants to build four adjacent rectangular pens. $130yd$ of fence are available. The current idea is to build the pens such that each vertical side is $10yd$ long and each horizontal side is $10yd$ long.

  Is it possible to get a larger total area by adjusting the dimensions?

  Trial and error could give the answer here and students who do this might have a reasonable amount of number sense. Students should (after trial and error?) be encouraged to structure their approach and use functions. Possible motivation would be to foreshadow the idea of maximizing the area.

Reading Quiz.

Notable Homework Problems.

Many of the problems given here will have a "comeback" in the (more typical) form of an optimization problem. It can be helpful to assign a few "strands" that consider the same setup under different guises, that is in terms of increases and decreases here and later as optimizations.

- Exercise 13a, even though the "solution" is ultimately wrong, contains the solution to Exercise 1.
- Exercise 14 could be labeled as a "brainteaser", but the idea in this section is to think mathematically, which this problem requires.

Section 1.4. Quadratic Functions

Suggested Time. 1 class period.

Lecture/Presentation.

- Quadratic functions
  1. Are of the form $f(x) = ax^2 + bx + c$,
  2. Have parabolas as graphs,
  3. Standard form. Give some pictorial examples with formula $f(x) = a(x-h)^2 + k$, show the vertex in each picture,
  4. Note that $(x-h)^2$ is nonnegative, so all output values will lie on the same side of $y = k$ (above or below)
- Find the vertex of $f(x) = 2x^2 + 4x - 8$. Complete the square, leave room to work things out formally, too. Possibly first work it out with the numbers, then go through what is written, replacing the numbers with the appropriate coefficients.
- Find the zeroes of $f(x) = 2x^2 + 4x - 8$. Where is the vertex in relation to the zeroes?

  Possibly first work it out with the numbers, then go through what is written, replacing the numbers with the appropriate coefficients. This would provide the quadratic formula.
- Show quadratic formula and its relation to the vertex formula.
- Sketch the graph of $f(x) = 2x^2 + 4x - 8$
- Introduce the idea of optimization. Stronger tools for finding maxima and minima are available through calculus.

Group Work/Examples.

- Explain via shifting results why the graph of $f(x) = (x+2)^2 - 4$ is a parabola with vertex at $(-2, -4)$.
- Sketch the graph of $f(x) = -2(x-1)^2 + 4$ (identify vertex, y-intercept and zeroes) by hand. Compare with standard parabola. What did the -2 do?
- Design a parabola that has $(2, 1)$ as its vertex and goes through $(4, 3)$.

  Could be used as a lead in to talk a little about curve fitting.
- A farmer wants to design eight adjacent pens in an egg-crate type arrangement (that is, there are two rows of four pens each that are adjacent to each other). 3000ft of fencing material are available. Determine the outside dimensions and the total area of the arrangement with the largest area.

Reading Quiz.

Notable Homework Problems.

- Exercise 5 is a variation on the typical "find a quadratic with properties ... " problems. The twist is that the constraints may not be satisfiable or lead to "pathological" cases (straight lines). The idea is to teach students to be on the lookout for bad data.

- Exercise 17 is a way to check if the solutions to a quadratic equation are actually correct. (I only know the German name: Vietascher Wurzelsatz.)

- Exercise 19 presents the quadratic formula as it is taught in many countries. You just have to divide by a first.

Section 1.5. Polynomials

Suggested Time. 1 class period.

Lecture/Presentation.

- Motivate the whole class period by wanting to sketch a given graph, say, $f(x) = x^4 - 10x^2 + 9 = \left(x^2 - 1\right)\left(x^2 - 9\right)$
- Introduce the long-term behavior. Identify leading term and long-term behavior in an example and in the leading example.
- Introduce division of polynomials. Motivate via the desire for a "complete" set of algebraic operations and the foreshadowing of asymptotic behavior of rational functions.

  Could use $P(x) = 4x^3 - x^2 + 3x - 1$ and $D(x) = 2x^2 + x - 1$ as example. Check the result, too.
- Remainder theorem. If $P(x)$ is divided by $(x - r)$, then $P(x) = Q(x)(x - r) + k$ and $P(r) = k$.

  Introduce synthetic division as a special case of division, motivated by this theorem. Could show how synthetic division arises out of long division.

  Because of the remainder theorem, synthetic division was (by hand) and is (in computer operating systems) a fast way to evaluate polynomials.

  Evaluate $P(x) = 4x^3 - x^2 + 3x - 1$ at $x = 5$ using synthetic division.
- Sketch the graph of $f(x) = x^4 - 10x^2 + 9 = \left(x^2 - 1\right)\left(x^2 - 9\right)$ using the procedure on page 37
- Construct a polynomial with given properties.

Group Work/Examples.

- Perform the division
  1. $P(x) = 2x^4 + 4x^3 - 3x^2 + x + 1$, $D(x) = 2x^2 - 2x + 1$
  2. $P(x) = x^5 - 3x^2 + 1$, $D(x) = x^2 + 1$
- Perform synthetic division for
  1. $P(x) = -4x^3 + x^2 - 1$, $D(x) = x - 3$
  2. $P(x) = 6x^4 - 12x^3 + x^2 - x - 2$, $D(x) = x - 2$
- Graph by hand (zeroes are possible, it factors) and extrema by zooming for $f(x) = 12 + 4x - 3x^2 - x^3$.

Reading Quiz.

Notable Homework Problems.

- Exercise 4 is a variation on the typical "find a polynomial with properties ... " problems. The twist is that the constraints may not be satisfiable. The idea is to teach students to be on the lookout for bad data. The request for an explanation teaches the verbalization of the conflicting information.

- Exercise 7 gives motivation for the terminology of "even" and "odd" functions.
- Exercises 8 and 9 show why synthetic division is correct.
- Exercises 8 and 10 show why synthetic division is computationally preferable to "regular evaluation".

Section 1.6. More on Zeroes of Polynomials

Suggested Time. 1 class period.

Lecture/Presentation.

- Discuss the nature of the Rational Zeroes Theorem. The implication goes in the "wrong" direction. The theorem says nothing if there are no rational zeroes.

  Example: $p(x) = x^3 - 2$ has no rational zeroes.

  Find all zeroes of $p(x) = 2x^3 - 5x^2 + x - 12$ (one rational, two complex).

- Fundamental theorem of algebra.

- By the fundamental theorem of algebra, every polynomial has a complex zero. Consequently, every polynomial can be factored.

  This does *not* mean that every polynomial can be "factored" as is perceived by many students. That is, the zeroes may be hard (or even impossible) to determine exactly and the high school guessing procedure will not work.

- Conjugate pairs theorem: If $P(x)$ is a polynomial with real coefficients and $P(a + ib) = 0$, then $P(a - ib) = 0$.

  Find all zeroes of $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$, given that $1 + 2i$ is a zero. (Four complex zeroes.)

- Design a polynomial with certain properties.

Group Work/Examples.

- Find all zeroes of

  1. $p(x) = 2x^4 - 3x^3 - 27x^2 + 62x - 24$ (all rational)
  2. $p(x) = 2x^3 + 3x^2 - 4x + 1$ (could multiply this one by x to get another small pattern covered.)
  3. $p(x) = 12x^4 - 29x^3 - 59x - 15$ (two rational, two complex)

- Find all zeroes of $P(x) = x^4 - 2x^3 + x^2 + 2x - 2$, given $1 + i$ is a zero.

Reading Quiz.

Notable Homework Problems.

- In Exercise 7 the reader can produce a proof of a standard fact from number theory (the square root of 2 is not rational) in a nonstandard fashion (using the rational zeroes theorem).

- Exercises 8 (Descartes' rule of signs) and 9 (the lower and upper bound rule) are two cute computational facts about polynomials that used to be part of the mainstream presentation. In the age of CASs they have lost much of their appeal. The lower and upper bound rule can still have theoretical value in showing that we can concentrate searches for zeroes on a finite interval. Descartes' rule of signs appears to be completely subsumed by the ability to "graph and count". If there is time, some historical discussion with students trying to imagine solving certain tasks without a CAS can be instructive.

Section 1.7. Computer Algebra Systems

Suggested Time. 1 class period.

Lecture/Presentation.

- A CAS is a tool, not a crutch. That is, we will use a CAS to enhance our abilities and solve problems that are impossible to solve by hand. Problems that can be solved by hand will still be solved by hand.
- Explain the ways in which use of a CAS can lead to incorrect results (user errors and systemic errors).
- Adjust window for $g(x) = -\frac{1}{4}x^4 + 4x^3 - 8x^2 + 2x + 1$

  Good advice: Even if the picture looks "good", move the window a bit unless you have sound theoretical reasons to assume the picture is right.
- Find the zeros of $g(x) = -\frac{1}{4}x^4 + 4x^3 - 8x^2 + 2x + 1$
- Find the approximate extrema of $g(x) = -\frac{1}{4}x^4 + 4x^3 - 8x^2 + 2x + 1$
- Explain the various rounding phenomena on the suggested activity.

Group Work/Examples.

- An activity as shown on the next page can be generated for the CAS of choice for the class. Possibly a 20 minute jigsaw with groups of 4; check if everyone has an assignment before starting.
  1. (5 min) Everyone does number 1, one accurately, three with rounding as indicated each adds in different order,
  2. (2 min) Each reports result, group finds explanation
  3. (8 min) Teams split up and everyone does one of 2-5 **individually**; 3 and 4 need a TI-82, 5 a TI$\leq$ 86, 2 works on any calculator, project pictures if necessary
  4. (5 min) Each reports the results and explanations are discussed and finalized
- Solve the equation
  1. $3x^4 - 2x^3 + x - 5 = 0.$
  2. $x^3 + 6 = 3x^2 + 2x$
  3. $3x + 5 = 7$
  4. $\sqrt[3]{x^3 + 8} - x = 2$

Reading Quiz.

Notable Homework Problems.

Surprising answers your calculator gives to easy tasks (TI-82 version)

While this worksheet is set up with the TI-82 in mind, similar phenomena can be observed on other brands of calculators also.

1. Warmup: The numbers 42.6, 545, 3.04, 5.06, 13.9, 92.4 all have three significant digits showing. The only difference is in where the decimal point lies. Please add these numbers

   (a) The usual way,

   (b) As follows: After you add two numbers round your answer to three digits before adding the next number.

   Try different orders of addition and explain your results.

2. Everyone knows that $2^{5x} = (2^x)^5$ and hence $2^{5x} - (2^x)^5 = 0$. Graph $f(x) = 2^{5x} - (2^x)^5$ in the window $[-10, 10] \times [-10, 10]$ (ZoomStandard on TI-calculators). Explain what you see.

3. Everyone knows that $f(x) = (x-1)^2 + 1$ is a parabola with vertex at $(1, 1)$. Graph $f(x) = (x-1)^2 + 1$ in the window $[.99995, 1.00005] \times [.\underbrace{999999999}_{9}5, 1.\underbrace{00000000}_{8}5]$. Explain what you see.

4. Everyone knows that $f(x) = \sin(96x)$ is a sine function that oscillates 96 times every 2π. Graph $f(x) = \sin(96x)$ in the window $[-2\pi, 2\pi] \times [-2, 2]$. Explain what you see. (Try $\sin(94x)$, too.)

5. (Only on TI-$\leq$ 86.) Everyone knows that $f(x) = \frac{1}{2}x$ is a straight line with slope $\frac{1}{2}$. Enter the expression `1/2X` as function y_1 into your calculator and graph it in the standard window. Explain what you see.

6. Graph the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. Compare what you see with what you should see.

Explanations for
"Surprising answers your calculator gives to easy tasks"

1 The rounding after every addition affects your answer. The closest one can get to the exact answer with this "rounded addition" is by first sorting the numbers from smallest to largest and then adding starting with the smallest number and adding the smallest remaining number to the intermediate result. In our example this actually gives the exact answer (if I set it up right), but in general that need not be the case. For example, 101+39.5 rounded to three digits will never be the exact answer as the exact answer has four digits.

2 Digital calculators only store a finite number of digits for any number they compute. Beyond these digits the stored number becomes random. A computation such as the subtraction of two numbers that are almost equal erases the first few digits of the numbers (for example, consider that 1567-1562 = 5 and the first three digits are erased). In an exact computation this is of no consequence, since we can easily retrieve the remaining digits. When an arithmetic only carries a finite number of places, however, we already have seen above that caution is necessary.

In a digital calculator, places that cancel each other in a subtraction are lost. In an extreme case, like our example, the numbers the calculator computes agree in all digits in which the calculator is exact. Thus all correct digits are "wiped out" and what remains is the random part. By the time x goes beyond 9, the numbers computed have more than 10 digits and the random part thus begins to have appreciable magnitude on our scale.

3 This effect is the visualization of the fact that calculators only store a finite number of digits. The horizontal bars are the values that the calculator can compute, the jumps between them represent the difference between two neighboring values that the calculator understands. The numbers between these values *do not exist for the calculator*.

Extension activity. Find a window in which the calculator graphs a horizontal line.

4 This effect shows that while the calculator plots lots of points, these points might be in the wrong place. Here the evaluations are spaced in such a way that the respective next point to be plotted is not just a small space away from the previous point, but *one period plus* a small space. The many oscillations that we expect are hidden between the points that the calculator "sees".

5 This one is specific to TI-calculators (all the above can be seen on any graphing calculator, provided you find the right window). Multiplication for which the multiplication sign is left out binds stronger than some other operations (consult the manual for the exact data). Thus the input `1/2X` is read as $\frac{1}{2x}$ by the calculator, which is what you see in the picture.

6 The function cannot be evaluated at 1, but since the calculator in most windows does not sample at 1, it looks like it can be evaluated there. (This can be used as a lead-in to limits.)

Section 1.8. More on Modeling – Optimization

Suggested Time. 1 class period.

Lecture/Presentation.

- Polya's (semi-)structured approach to problem solving. "Define, plan, execute, evaluate."
- Connection to "real life".
- Review that we have done modeling twice already, we just have better tools now.
- Use of CAS for finding extrema will later be replaced by exact methods of calculus that also apply to more symbolic problems.

Group Work/Examples.

- A rectangular box is to be made from a 10in $\times$ 15in piece of sheet metal by cutting out equally sized squares at the corners and folding up the sides. (Shop class exercise? Teaches cutting and folding of metal with all its little pitfalls.) Find the dimensions of the box with the largest volume.

  Use CAS to find the maximum. Convert the point we find into the answer.
- A rectangular box whose base is twice as wide as it is long is supposed to have volume 1000in^3. Find the dimensions of the box with the smallest surface area.
- If not assigned otherwise, Project 1.9.3 is an authentic project that can show the utilization of optimization processes in industry. (Should mention that the pictures are for comparison only. Some students tried to get the answers from the technical specs instead of following the instructions.)

  Could conclude the discussion by showing that CAS does not help if the thicknesses of sides and top are unknown (which is the first step in a completely new design process).

  Could use any of the optimization projects in Section 1.9 in similar fashion.

Reading Quiz.

Notable Homework Problems.

Section 2.1. If-Then-Statements (Implications)

Suggested Time. 2 class periods, roughly one for the logical definitions and piecewise defined functions and roughly one for the inverse functions.

Lecture/Presentation.

- Could tell the introductory story as motivation.
- Motivation for CS oriented people. There are three basic components to programming, which are Input/output, decision structures and iteration structures. Our investigation of implications provides the foundation for decision structures from an abstract point-of-view.

  This fits right in with piecewise definitions as a start. Remember to introduce the absolute value function here.
- Turn the decision into the logical statement (which is a decision if the conclusion applies).

  Example: **If** the triangle in the picture is a right triangle, **then** $a^2 + b^2 = c^2$. (First only record $a^2 + b^2 = c^2$ as Pythagoras' law.)

  Draw two triangles with standard notation (define). One right triangle, one not right. Give lengths of sides $a =$, $b =$, $c =$.
- Conditional statements: "If P, then Q." P is called the hypothesis and Q is called the conclusion.
- Mention the "if ... then ... else" construct in programming.
- A true conditional statement says the following. Provided the hypothesis is true, the conclusion is guaranteed to be true also. Nothing can be said if the hypothesis is false or if the conditional statement is false.

  WARNINGS.

  1. Be careful with the hypothesis. It may be stronger than what you can assume, which could put the conclusion in jeopardy.
  2. Just because you cannot find a counterexample does not make the statement true.
- Examples (audience: which are right or wrong?)

  1. If there is lightning in the area, then the football game will be cancelled.
  2. If $a > 1$ then $f(x) = a^x$ is increasing.
  3. If $|x| < 6$, then $x < 6$
  4. If $ab = 0$, then $a = 0$ and $b = 0$
  5. If $a < b$, then $ca < cb$
  6. If $\frac{a^x}{b^y}$ is a quotient of powers, then we simplify by subtracting exponents.
- To show a conditional statement is false we only need *one* counterexample (find counterexamples for false statements)

  To show a conditional statement is true, we need a proof (and that is what mathematicians do)

- Exercises 1 and 2 in Section 3.3 can be used as examples of chains of implications ("proofs").

- Look at the above examples, find which are true, give counterexamples for those that are false.

  Formulate the correct statement(s) for the one on quotients of powers (this can be a common mistake, maybe caused by not caring about the hypothesis)

- The converse of "if P then Q" is "if Q then P"

- Formulate the converses of the examples.

  Which of the converses of the example problems are true or false? Give counterexamples for false ones. Point out that a statement and its converse are not equivalent.

- Biconditional statements: "P if and only if Q" means "If P, then Q" AND "If Q then P."

- Example: $ab = 0$ if and only if $a = 0$ or $b = 0$

  Re-write the above examples for which statement and converse are true as biconditionals.

- Contrapositive. The contrapositive of "If P, then Q" is "If not Q, then not P".

- The contrapositive of a conditional statement is logically equivalent to the original conditional statement, i.e., they are either both true or both false

- The contrapositive and the converse are not the same thing.

- Example: Contrapositive of "If it is snowing then it is cold." is true, the converse is not.

  ——————— use of implications in precalculus starts here ——————

- Lead-in to inverses, if the law of cosines is available. Good authenticity if many students are in an engineering class that does some surveying.

  Given a triangle in standard notation with $a = 3$, $b = 5$, $c = 7$, what is the angle opposite c?

- If law of cosines is not available, could use how social security numbers are traced back to people for the allocation of benefits. Breakdowns in the one-to-one structure are problematic (identity theft).

- Define one-to-one functions.

  Show horizontal line test.

- Use contrapositive to get a workable formulation.

- Determine if the function is one-to-one.

  1. $f(x) = \sqrt{3x + 1}$
  2. $f(x) = (5x - 2)^2$

- Going from function to its inverse means switching x and y. Graphically this is a reflection across $y = x$.

- Find the inverse of $f(x) = 3 + \frac{5}{x}$ (presented, sketch graphs)
- Talk about domain restrictions
  1. If we restrict $f(x) = 2x^2 + 1$ in 3 to $x \geq 0$, then the inverse is $g(x) = \sqrt{\frac{x-1}{2}}$ with $x \geq 1$,
  2. Square root and square are inverses when restricted to nonnegative numbers

Group Work/Examples.

- For the function $f(x) = \begin{cases} 3x+1; & \text{for } x < 0, \\ 4; & \text{for } 0 < x < 3, \\ x^2 - 15; & \text{for } x \geq 3. \end{cases}$ find $f(2)$, $f(5)$, $f(2.9)$, $f(-3)$, $f(3)$, $f(0)$.
- For the following conditional statements decide if they are true or false. **REPEAT. Not all of these statements are true.** For the false statements give a counterexample. For the true statements formulate the converse and the contrapositive.
  1. If it is Thursday, 10-10-2002, then I am supposed to take a math test today.
     (Put the date of your next, usually the first, test in here as a casual reminder for students.)
  2. If a, b, c are the sides of a triangle, then $c^2 \leq a^2 + b^2$.
  3. If a, b, c are the sides of a triangle and A, B, C are their respective opposite angles, then $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$.
  4. If $a > 0$, then $f(x) = a^x$ is increasing.
  5. If $0 < a < 1$, then $a^2 < a$.
  6. If $0 < a < 1$, then $\sqrt{a} < a$.
  7. If $3^x 2^y$ is a given product, then we can simplify it to 6^{x+y}.
  8. If the angle α is in the third quadrant with $\cos(\alpha) = -\frac{1}{3}$, then $\sin(\alpha) = -\frac{8}{9}$.
- Find symbolically which (if any) of the following functions is 1-1. Lead in by showing how it is easier to work with the contrapositive of the definition of a one-to-one function, since it is easier to work with equations than with inequalities.
  1. $f(x) = 3x + 2$,
  2. $f(x) = x^3 + 3$,
  3. $f(x) = 3x^2 - 1$.
- Check which of the following are inverses of each other
  1. $f(x) = 3x + 5$, $g(x) = \frac{x-5}{3}$
  2. $f(x) = \frac{3}{x^3+1}$, $g(x) = \sqrt[3]{\frac{3}{x} - 1}$

3. $f(x) = 2x^4 + 1$, $g(x) = \sqrt[4]{\dfrac{x-1}{2}}$
4. $f(x) = x^2$, $g(x) = \sqrt{x}$

- Find the inverse of

1. $f(x) = 3x^3 + 2$
2. $f(x) = \dfrac{x+4}{3x}$
3. $f(x) = 2x^2 + 1$

Reading Quiz.

Notable Homework Problems.

- It is possible to start an integrated precalculus-calculus class or a calculus class with Module 2 and use Module 1 for references and student review outside of class. To enforce this approach, Exercise 16 can be assigned on the first class day. The colleague who created this approach (Galen Turner) was once asked by a student "Was last night's homework a logic homework or a way to make us review algebra"? His answer of course was "Yes" (to the inclusive or).

  In this approach, slightly extended coverage of Module 2 can give a thorough review of functions from a new point-of-view. For example, Exercise 16a can be used to lead into modeling and optimization after the definitions have been analyzed by the students.

- Exercise 6 should be very realistic to any student who frequently calls home. (Or if not to the student, then to the student's parents.)

Section 2.2. Quantified Statements

Suggested Time. $\frac{1}{2}$ class periods.

Lecture/Presentation.

- Introduce the universal quantification as an abbreviation for an "if-then" statement. For example "For all positive numbers a and b the product ab is positive." is much shorter than "If a is a positive real number, then if b is a positive real number, then the product ab is positive."
- Could also use the definition of inverse functions as an introduction.
- For a universal statement to be false we only need one counterexample.

  Decide which are true and which are false. (These also reenforce the need for precision.)

  1. "For all real numbers a, b we have that $a^2 + b^2 \geq 0$."
  2. "For all $x > 0$ we have $x^2 > x$."

- Note that any kinds of "formulas" in mathematics, such as $a^2 - b^2 = (a + b)(a - b)$, usually are quantified statements, in this case "For all real numbers a and b we have that $a^2 - b^2 = (a + b)(a - b)$."
- Emphasize that formulas that students are not sure of can be debunked by finding a counterexample. Typical problem cases are wrong attempts like $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.
- Introduce existential quantification via statements that guarantee existence of certain objects (square roots, zeroes of polynomials, fundamental theorem of algebra "If P is a polynomial with complex coefficients, then there exists a complex number z such that $P(z) = 0$.")
- Problems 5 and 17d (solution) in Section 1.2 show that for an existence statement to be true we only need one example.

  Decide which ones are true and which are false.

  1. "There is a function f such that $f(1) = f(2) = f(3) = 0$."
  2. "There is a real number x such that $x^2 + 1 = 0$."

- Examples of nested statements.

  1. "For all positive real numbers x there is a positive real number s such that $s^2 = x$."
  2. "For all nonzero numbers x there is a nonzero number r such that $rx = 1$."

- Motivation of the building of the definition of $a^{\frac{p}{q}}$ for computing oriented people. Imagine you have a bare-bones programming language and you have to construct higher functions (such as powers) yourself.

  This is frequently an exercise in CS classes taught in close curricular proximity to precalculus/calculus I.

- Define how a "constant" works in mathematics.

Group Work/Examples.

Reading Quiz.

Notable Homework Problems.

Section 2.3. AND, OR and other connectives

Suggested Time. 1 class period

Lecture/Presentation.

- Motivation for computing and EE-oriented students. AND, OR and NOT are at the heart of every digital circuit. In fact, (cf. page 84) these operations have to be constructed from the NAND operation.

  It would not be inappropriate to make up a test question using logic gates if all or most students are in EE and CS and know the standard notation for logic gates.

  Another introduction is through the legalese of scholarship requirements. "Students qualify for the scholarship by ... OR ... OR ... " (inclusive or) and "To maintain the scholarship you must ... AND ... AND ... "

- Introduce the logical connectives and their truth tables. Pay special attention to OR and $\Rightarrow$.

- Set up the truth table of $\neg(P \text{ AND } Q)$

- Solve some inequalities, say $|3x - 1| < 4$ and $|-x + 9| \geq 1$ to reconnect AND and OR to familiar precalculus work.

  Do this before discussing $\Rightarrow$ and proofs by contradiction.

  This can be used especially to illustrate that OR should be inclusive.

- Give an example of a truth table with three variables, say $(P \text{ OR } Q) \text{ AND } R$.

- Then introduce $\Rightarrow$. Explain why the main focus is on excluding $1 \Rightarrow 0$.

  Then they do a truth table with three that involves $\Rightarrow$.

- Motivation for proofs by contradiction aside from being a standard proof technique used in math and CS. One can dismiss incorrect arguments in any walk of life (including legal battles) with it. ("Let's just pretend it's true and let's see what would happen. You'll see nothing good could have come from it.")

  Real life example for CS students. Testing code. Any program you test, you feed data under the assumption that the program performs correctly. Then you compare the expected output with the actual output. If there is a mismatch, you have work to do.

- Earlier examples of possible proofs by contradiction. The statement "It is the only function that is even and odd." in Exercise 17d in Section 1.2. Exercise 7b in Section 1.6.

  CS example of using contradiction. If the code is true, then with this input we should get an output value of 2. The output is not 2, so there must be a problem with the code.

- Solve an equation by using a chain of implications to show where an extraneous solution is introduced. (Re-emphasize that false can imply true.)

  Example $\sqrt{x+1} + 1 = x$

Group Work/Examples.

- Solve the inequality $|3x - 2| < 5$
- Find the truth table of
  1. $(P \text{ AND } Q) \Rightarrow R$
  2. P AND $(\neg Q)$ (to prepare for negation)
  3. $(\neg P)$ OR $(\neg Q)$ (to prepare for negation)
- Solve $\dfrac{x}{2} + \dfrac{3}{x} \geq 1$
- Solve $\sqrt{x+1} - x = -5$

Reading Quiz.

Notable Homework Problems.

- The distributivity of or over and in problem 5a is a standard result for logic and also a natural counterpart to Example 2.3.3.
- Exercise 10d might help prevent any kind of nonsensical "arguments" at the end of the term.

Section 2.4. Negation of a Statement

Suggested Time. 1 class period

Lecture/Presentation.

- Negations are useful for proofs by contradiction, as well as to truly understand a statement (by determining what it means that the statement is not true).
- Explain how to negate some of the logical connectives via truth tables.

  Make some of these explanations group work.

  Connect the negation of $\Rightarrow$ to the structure of proofs by contradiction.
- Show how to negate nested quantified statements (work the negation from the outside in).

  Negate "For every real number x there exists a real number r such that $r^2 = x$."

  Breakup:

  For every real number x
  there exists a real number r such that
  $r^2 = x$.

  Now work the negation in from the outside.

Group Work/Examples.

- Find the negation of the statement.
  - f is a given function.
    "For all real numbers x we have that f is continuous at x."
  - f is a given function
    "For all real numbers x there is a real number r such that $f(x) \cdot r = 1$."
  - f is a given function and a is a given point.
    "For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all x we have that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$."
  - f is a function.
    "f is increasing"
    (Check how many say "decreasing". Could use the sine function as an example that is neither.)
- State what it means that a function is neither increasing nor decreasing.

  (Could motivate this by noting that the sine function is neither increasing nor decreasing and we want to know exactly how to say that.)
- Find the contrapositive of the statement.
  - "If $a < b$, then $-a > -b$."
  - "If $a > 0$ and $b > 0$, then $ab > 0$."

- "If $a = 0$ or $b = 0$, then $ab = 0$."
- "If f is differentiable at x, then f is continuous at x."
- "If $a < b$ and $ab \neq 0$, then $\frac{1}{a} > \frac{1}{b}$."

Reading Quiz.

Notable Homework Problems.

- Exercises 1 and 2 prove the DeMorgan laws, which are fundamental.

Section 3.1. Measuring Angles

Suggested Time. $\frac{1}{2}$ class period.

Lecture/Presentation.

- Define angles using all appropriate terminology. Use a picture.
- For measurement of angles we have two systems: Degrees (full circle cut into 360 equal slices of size 1°) and Radian measure (1 radian is the angle that intercepts an arc of length 1 on a unit circle).
- Degree measure has religious origins. (The Babylonians thought the year has 360 days. Good thing, too. Could you imagine working with a 365° circle?)

  Thus it is customary but also somewhat arbitrary. Indeed there was a movement to give a circle 400 degrees ("new degrees", was available as "grad" on calculators).
- "Radian" is not a unit. Radian measure is dimensionless.
- Convert $20^\circ 46' 14''$ to decimal degrees, convert 67.5924° to degrees, minutes, seconds.
- In radians: angle = length of the arc cut out of a circle of given radius divided by the radius

  (It can be proved that the number is independent of the radius.)
- Show conversion between degrees and radians. Idea for conversion (so mathematicians and engineers/scientists can talk): 1 revolution is 360° which is also 2π.

  Convert 84° to radians.
- Conversion between the measures is no different than any other unit conversion process.

  Could formalize it by changing each unit into its equal measure in another unit, say $23' = 23 \cdot 1' = 23 \cdot \frac{1}{60}^\circ$ ...
- Angles in standard position, quadrants, coterminal angles.
- If the central angle of an arc on a circle of radius r is θ, then the arc on the circle is of length $s = r\theta$.

  Big advantage for radian measure. Rulers are straight, so they are not quite appropriate to measure arcs.
- Standard position, quadrants, mathematically positive direction

Group Work/Examples.

- Step 1: find examples of measurement processes

  Step 2: try to define what a measurement is. [Comparison with a standard object (a meter, a kilogram, etc.)]

  Good connection to ENGR and science here.

- Convert $45°25'10''$, $98°20'25''$, $3\pi/2$, 3 to the respective other system of measurement,
- Sketch angles of $-2\pi/3$ and $450°$.
- How long is an arc on a circle of radius 5m with central angle $80°$?

  Could dress this up using the turning radius of a car around a hairpin turn, asking how far the car goes.
- Problem for informal groups. My car tires have a radius of 20cm and are currently performing 6000 rpm. How fast am I going?

  (This is very fast. The problem is made up. Use this to explain that the rpm measure in the car is *not* the measure of how many rpm the wheels perform.)

  Choose more realistic numbers if this appears to have too much "stuff" in it.

  Problem motivates the idea that angular speed can be translated into linear speed.

  Show angular velocity $\omega = \frac{\theta}{t}$ and derive $v = \omega r$
- How fast do tires with 8in radius rotate when the car is going 65mph?

Reading Quiz.

Notable Homework Problems.

- Exercises 12 and 13 explore the mechanics of hard disk drives.
- Exercise 15 explores the behavior of cars in a curve. This would be interesting to car enthusiasts and mechanical engineering student.

Section 3.2. Defining the Trigonometric Functions

Suggested Time. $\frac{1}{2} - 1$ class period.

Lecture/Presentation.

- Draw angle in standard position (quadrant 1), draw triangle and let audience help with the definition of the trig functions by asking them to recall what they did with right triangles.

  (Note to those who have not seen it, that we are building things from scratch. They will not lack in preparation through the class, but they might lack experience.)

- Picture: Define the three trigonometric functions through the usual circle picture. (Note that there are properties on similar triangles that show the definition is sound.)

  Note that as we go beyond $\frac{\pi}{2}$, we always draw a vertical line to the x-axis to get our triangle. (This will always be our reference, have picture in text.)

  Why do we want to define these functions for angles other than angles that we find in a right triangle? Rotational motion is repetitive. Thus the trigonometric functions will exhibit repetition patterns also, which can be used to model repeating natural phenomena (temperature, tides, etc.)

- Note that $\tan(\theta)$ is not defined for $\theta = \frac{\pi}{2} + n\pi$.

- Elementary identities: $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, $\sin^2(\theta) + \cos^2(\theta) = 1$

- Find the values of the trigonometric functions for an angle in standard position with terminal side going through (2, 3).

  This one recalls Pythagoras. (Note the theorem for those that have not had trig.)

  Remark that there are infinitely many such angles, and that the trig functions are unaffected by a change by 2π,

- Show the different signs of sin and cos in the different quadrants, could also give a new spin on the ASTC acronym: All Students are Terrified of Calculus, but we are not ;)

- If $\tan(\theta) = -\frac{2}{5}$ and θ is in quadrant II, what are the values of the trig functions at θ?

  (Do not use a calculator. Or use the fact that arctan gives the wrong angle anyway, which might be interesting to show.)

- Tangent is important because the slope of a line is the tangent of its angle of ascent.

- Mention $\sec(x)$, $\csc(x)$, $\cot(x)$.

- In the unit circle definition, the "missing side" is always vertical and ends on the x-axis.

- For most purposes it is enough to be *very* comfortable with sine and cosine.

Group Work/Examples.

- $\sin(\theta) = -.3$, θ in quadrant III. Find the trigonometric functions.
- If $\sin(\theta) = \frac{2}{3}$, θ is in quadrant II, find the values of the trigonometric functions at θ.
- Let θ be an angle in the second quadrant with $\tan(\theta) = 2$. Either
  - Find $\cos(\theta)$, or
  - Explain why such an angle does not exist.
- Find the equation of the line with a $55°$ angle of ascent that goes through the point $(-5, 2)$.

Reading Quiz.

Notable Homework Problems.

- Problems 9, 10, 11 and 12 all relate to the angle of ascent of a line/road. Not terribly fancy, but the analysis will force the student to understand the tangent function.

Section 3.3. Applications of Triangles

Suggested Time. $\frac{1}{2} - 1$ class.

Lecture/Presentation.

- Present an estimate such as Example 3.3.1 (or the example itself) as motivation.
- Present a problem similar to Exercise 1 or 2.
- An airplane has a line-of-sight distance of 3 miles to a control tower and the line of sight makes an angle of $10°$ with the horizontal ground. How high is the airplane flying?
- Two rangers are on a missing person search in the desert. (Robotic vehicles exploring a landscape such as Mars or being on a hunt for explosives might be more "realistic".) They maintain $200yd$ distance between them at all times. The first ranger spots something in a direction that makes a $34°$ angle with the line of sight to the other ranger. The second ranger spots the same thing, but at an angle of $58°$ to the line of sight.

  Which ranger is closer to the object? What are the distances involved?
- Derive the area formula for a regular hexagon.

Reading Quiz.

Notable Homework Problems.

- Problem 9 "Treegonometry" is a measurement that the author actually did once around his own house. (The tree did not need to go.)

  Another nice feature is that "the answer" actually is a recommendation, not a number.
- Problem 10 is a fairly well-known brainteaser.

Section 3.4. The Graphs of Sine, Cosine and Tangent

Suggested Time. $\frac{1}{2} - 1$ class period.

Lecture/Presentation.

- Show the construction of the graphs of sine, cosine, tangent. Sketch a graph of $\sin(x)$ by sideways projection from the circle, Use the standard angles as help. (Essentially plotting points.)

  (The graphs of the sine and cosine functions are sideways projections of rotational motion.)

- Could use the construction to motivate the definition of periodicity.
- Give domain, range, intercepts, period for sine
- Show the sideways projection of the circle on CAS.
- Give domain, range, intercepts, period for cosine
- Discuss the graph of the tangent function as quotient of sine and cosine, Give domain, range, intercepts, period for tangent
- Prove negative angle identities and/or show them graphically. (If they have not been part of the presentation of the properties of sine and cosine already.)
- Prove cofunction identities and/or show them graphically.
- Explain graphical meaning of cofunction and negative angle identities.
- Incorrect proposed identities can be dismissed by finding *one* point where they don't work. (See Exercise 3. This foreshadows or reinforces the logic part.)
- Define frequency
- Prove the values of the trigonometric function at the standard angles.
- Values of the trigonometric functions are convenient mainly in calculus classes. They are can easily be remembered using Figures 3.38 and 3.39.
- Value of trig function at an angle: Magnitude determined via reference angle, sign by quadrant.

  Find the values of the trig functions at $135°$ and at $\frac{7\pi}{6}$ (draw picture).

Group Work/Examples.

- One partner moves pencil back and forth horizontally, the other pulls out the paper at uniform speed. This produces the graph of a harmonic oscillation.
- Classroom activity on graphs on page ACT-10.
- Classroom activity on graphs on page ACT-11
- Sketch the graph of $\sin(x)$ in $[-5\pi, 4\pi]$.
- Some recognition problems like Exercise 2.

- Find the values of the trig functions at $300°$ and at $\frac{3\pi}{4}$.

  Could also ask for $\sec\left(\dfrac{\pi}{3}\right)$ to see if students remember how to deal with the "nuisances".

- Some simplifications and equations that can be checked graphically and proved with the identities we have at hand.

Reading Quiz.

Notable Homework Problems.

- Exercise 7 foreshadows Taylor polynomials.
- Exercise 8 is intended to show the manifold connections between sine and cosine via horizontal shifts and flips. Good reinforcement of the negative angle and cofunction identities, which are our only tool at that time.
- Exercise 9 proves another "well known fact from algebra" with methods accessible to the student.
- The unit circle in Exercise 17 and the table in Exercise 18 are often given in texts. Since I want to avoid excessive, unnecessary memorization, they are for the student to fill out. Memorization is at the student's discretion, plus memory will be aided by having created the graphic or table.

Section 3.5. Amplitudes, Periods/Frequencies and Phase Shifts

Suggested Time. 1 class period

Lecture/Presentation.

- Not every phenomenon that is modeled with a sine wave will have height 1, period 2π and no phase shift.

  Example: Voltage we draw from a regular wall socket.

  Ask for its "height" and its frequency, translate frequency to period.

- Rotational and wave motions are distinguished by their amplitude, period (frequency), and phase shift

- Possibly also a qualitative discussion of predator-prey modeling.

- Amplitude: Vertical stretch factor: the a in $y = a\sin(x)$. Remark that this way we can give the sine any "height" we want.

  Sketch one or two sines with different amplitudes.

- How do we speed up the oscillation? Need to multiply the argument with a number. Example. For what value of x does $2x$ equal 2π? Essentially multiplication with a factor "speeds up time" or "slows it down".

  Period of $y = a\sin(bx)$ is from $bx = 0$ to $bx = 2\pi$. Thus the period is $\dfrac{2\pi}{|b|}$.

  Sketch some sine functions with different periods. If the CAS is used, predict what the picture is to look like before we graph.

- Finally we will not always have it that a wave phenomenon starts with $f(0) = 0$. To model this we have the phase shift, which is very important when several phenomena are observed that are out of synch.

- $y = a\sin(bx + c) = a\sin\left(b\left(x + \dfrac{c}{b}\right)\right)$ is a sine wave with amplitude a, period $\dfrac{2\pi}{b}$, shifted to the left by $\dfrac{c}{b}$ units.

- To emphasize the importance of phase shifts let student (graphically) add $\sin(x) + \sin(x)$ and $\sin(x) + \sin(x + \pi)$

- Sketch $y = 2\sin(3x - 1)$ and develop the procedure.

  1. Find the phase shift and mark it.
  2. Go one period to the right and to the left.
  3. Cut the periods in four equal pieces.
  4. Scale the y-axis.
  5. Sketch the graph.

  Check with CAS (possibly overlay the OH image with the picture on the board.)

- Place to scale tangent and cotangent graphs: Midway between the zero and the singularity the value is originally 1 and in a scaled setting equal to the amplitude

- The graphing procedure works equally well for all trig functions.
- Read periods and amplitudes off some graphs. Idea: Students will get pictures of oscillations (measurements in electrical engineering for example) and will need to turn it into a formula.

Group Work/Examples.

- Classroom activity on page ACT-12.
- Sketch (without CAS)

  1. $y(x) = \frac{1}{2}\sin(4x + 2\pi)$
  2. $y(x) = 4\cos\left(\pi x + \frac{\pi}{2}\right)$
  3. $y(x) = \frac{1}{2}\tan\left(\pi x + \frac{\pi}{2}\right)$

- Could give one of the nuisance functions to sketch to determine if students remember how to overcome the nuisance.

Reading Quiz.

Notable Homework Problems.

- Exercise 4 foreshadows some trigonometric identities and reenforces the utility of a graphical approach and pattern recognition.
- Exercise 6 shows how pictures of trigonometric functions are used in frequency measurements.

Section 3.6. Inverse Trigonometric Functions

Suggested Time. $\frac{1}{2}$ class period.

Lecture/Presentation.

- To sight on a target that is 5000m away and 3000m above the ground, at what angle do we need to set a telescope?

  We know that the tangent of the angle will be $\frac{3}{5}$, but what will the angle be? Need to go from the output to the input. That is, we need the inverse function of the trigonometric functions.

- Start with sine. Problem: The sine function is not one-to-one.
- Must restrict the domain where it is invertible.
- Define the **inverse sine** or **arcsine** function by $y = \sin^{-1}(x) = \arcsin(x)$ if and only if $x = \sin(y)$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
- Find arcsin (0), they: $\arcsin\left(\frac{1}{2}\right)$, arcsin (2),
- Define the **inverse cosine** or **arccosine** function by $y = \cos^{-1}(x) = \arccos(x)$ if and only if $x = \cos(y)$ for $0 \le y \le \pi$.
- Define the **inverse tangent** or **arctangent** function by $y = \tan^{-1}(x) = \arctan(x)$ if and only if $x = \tan(y)$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$.

  But $\sin^{-1}(\sin(x)) = x$ only for all x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **NOT** for all real numbers.

- Similar statements apply to the other functions.
- Find $\sin(\arcsin(.34))$, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$, $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$, $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$, $\cos\left(\arccos\left(\frac{1}{2}\right)\right)$, $\arctan\left(\tan\left(\frac{5\pi}{6}\right)\right)$,

  Traps: some invtrig(trig) combinations will give a different angle. Need to explain this with pictures.

Group Work/Examples.

- Find arccos (0), arctan (1), $\arcsin\left(\frac{1}{2}\sqrt{3}\right)$, $\arccos\left(\frac{1}{2}\right)$, $\arccos\left(\frac{1}{2}\sqrt{2}\right)$, $\arctan\left(\sqrt{3}\right)$, arctan (9),
- $\sin(\arccos(.9))$, $\cos(\arcsin(-.75))$, $\tan(\arcsin(-.33))$,

Reading Quiz.

Notable Homework Problems.

Section 4.1. Proofs by Induction

Suggested Time. 1-2 class periods

Lecture/Presentation.

- How do you prove a statement that is supposed to be true for all integers? Checking every single one is out of the question. Approaches via functions are limited because specific statements about integers may not generalize to functions of a continuous variable.

- Possible lead-ins are Gauss' summation formula for integers or (if derivatives already covered) formulas for n^{th} derivatives.

- Motivation for induction proofs:

  1. $\dfrac{k}{2}(k+1)$ and $\dfrac{3k^2}{2} - \dfrac{7k}{2} + 3$ are equal for $k = 1, 2$
  2. $\dfrac{k}{6}(k+1)(2k+1)$ and $\dfrac{5}{2}k^2 - \dfrac{7}{2}k + 2$ are equal for $k = 1, 2, 3$
  3. $\dfrac{1}{4}k^2(k+1)^2$ and $\dfrac{k^4}{4} + \dfrac{3k^3}{2} - \dfrac{183k^2}{4} + 369k - 324$ are equal for $k = 1, 2, 3$

  Hence to prove a formula for all k, we need more than equality early on.

- Also note that induction proofs of summation formulas sharpen algebra skills.

- Discuss the principle of induction. Domino and image-within-image comparisons.

- Possible way to avoid the impression that we use the conclusion to prove the conclusion: State the result to be proved as $P(n)$ and state the proof in terms of $P(k) \Rightarrow P(k+1)$.

- Demonstrate with

  1. $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k}{6}(k+1)(2k+1)$
  2. $1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^k} = 2 - \dfrac{1}{2^k}$ (Students may see a pattern here. Once you bring the $1/2^k$ over to the right, the carry-overs in base 2 arithmetic do the trick. This would just give reassurance that induction do the right thing.)
  3. If derivatives are already available, show that if $f(x) = x^{-\frac{1}{3}}$, then $f^{(k)}(x) = (-1)^k \dfrac{1}{3}\dfrac{4}{3} \cdots \dfrac{3k-2}{3} x^{-\frac{3k+1}{3}}$ (also prepares for Taylor polynomials)

Group Work/Examples.

- Prove by induction.

  – $2 + 4 + \cdots + 2k = k(k+1)$

  – $(1+x)\left(1+x^2\right)\cdots\left(1+x^{2^k}\right) = \dfrac{1 - x^{2^{k+1}}}{1-x}$

  - For all $k \geq 5$: $2^k \geq k^2$
  - (If differentiation is already known.) If $f(x) = \dfrac{1}{\sqrt{1-x}}$, then for $k \geq 1$ we have
    $f^{(k)}(x) = \dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k}(1-x)^{-\frac{2k+1}{2}}$

Notable Homework Problems.

- We can prove summation formulas for the powers of the first k integers, but students may rightly ask how people had the idea for such formulas. Exercise 6 shows how to create the summation formulas. This exercise should make for a nice project. As a homework problem it is definitely on the "high end".

  (Of course, once we create summation formulas in this fashion, they are already proved. Induction would only be a way to double check our algebra.)

Section 4.2. Summation Notation.

Suggested Time. $\frac{1}{2}$ class period.

Lecture/Presentation.

- Motivation for summation notation.
  - Want to have more compact, precise notation for processes that require summation (polynomials of arbitrary degree, Riemann sums, summation formulas we may have seen in induction, etc.)
  - Only with this type of notation will we be able to let a computer evaluate a sum. (Magical three dots just don't do it.)
- Major benefit: Precision in the argument.

Group Work/Examples.

- Evaluate the sum (by hand and/or with a CAS)
  - $\displaystyle\sum_{n=1}^{4} n^2$
  - $\displaystyle\sum_{n=2}^{6} \frac{1}{n^2} - \frac{1}{(n-1)^2}$
- Write the sum in summation notation
  - $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{20}$
  - $1 + 4 + 9 + 16 + 25 + 36 + \cdots + 729$

    The remaining problems are a preparation for Taylor polynomials.

    In each case the instruction should be that there are N terms in the sum. Alternatively, one could write down some more terms and handle this like the above problems.
  - $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
  - $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
- Prove the summation formula by induction
  - $\displaystyle\sum_{n=1}^{k} n^2 = \frac{1}{6}k(k+1)(2k+1)$
- Show by shifting indices that $\displaystyle\sum_{n=2}^{k} \frac{1}{n^2} - \frac{1}{(n-1)^2} = \frac{1}{k^2} - 1$

  (Telescoping sum.)

- Compute the limit $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{2}{n} \left(j \frac{2}{n} \right)^2$.

  (Quote the summation formulas in Example 4.2.4 and in Exercises 2a and 2b.)

Notable Homework Problems.

- Exercise 4h is the encoding of a sequence of coefficients that is needed to formulate Simpson's rule in summation notation.
- Exercise 5 requires students to compute limits of sums that are identified as Riemann sums in Exercise 2 in Section 12.2.
- Exercise 8 introduces binomial coefficients and the binomial theorem.

Section 4.3. Systems of Linear Equations

Suggested Time. 1 class period.

Lecture/Presentation.

- Motivate systems of linear equations
  - Geometrically (intersection of two lines, the higher dimensional stuff should be preserved for when lines and planes are discussed).
  - Via an application.
  - Via other uses in math (partial fraction decomposition, interpolation with a given type of curve, etc.)
- Terminology associated with systems of linear equations. Solution, equivalent systems, elementary row operations (switch, multiply by constant, add two equations and replace one with the sum)
- Show how to use the elimination method to solve a system of equations.
- Emphasize that solutions can be checked.
- Explain underdetermined systems (multiple solutions) and overconstrained systems (no solutions), possibly with intersections of lines.

  (Use examples that they do in groups as lead-in.)
- Show how to solve a system of equations on a CAS.

Group Work/Examples.

- All problems in this section are included in the activity on page ACT-13.
- Determine which of the given tuples of numbers solve the system of equations.

  1. $$\begin{array}{rcrcrcr} -x & + & 2y & + & 3z & = & 3 \\ x & + & 2y & - & 4z & = & -2 \\ 2x & + & 3y & + & z & = & 7 \end{array}, \qquad (4, -1, 1), (8, 1, 3)$$
     Neither solves.
  2. $$\begin{array}{rcrcrcr} -x & + & 2y & + & 4z & = & 3 \\ x & + & 2y & - & 4z & = & -2 \\ 2x & - & y & + & z & = & 0 \end{array}, \qquad (-1, -1, 1), \left(-\frac{1}{6}, \frac{1}{4}, \frac{7}{12}\right)$$
     $(-1, -1, 1)$ does not solve
  3. $$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & -7 \\ -x & - & 2y & + & z & = & -2 \\ & - & 7y & + & z & = & -11 \end{array}, \qquad (1, 2, 3), (-4, 1, -4)$$
     (Both solve.)

- Solve the system of equations using the elimination method.

1.
$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 4 \\ 2x & - & y & + & z & = & -2 \\ 2x & - & y & + & 2z & = & 0 \end{array},$$
solution: $(-4, -4, 2)$

2.
$$\begin{array}{rcrcrcr} 3x & - & y & + & 2z & = & 4 \\ 2x & + & 4y & + & z & = & -2 \\ 2x & - & y & + & 2z & = & 0 \end{array},$$
solution: $\left(4, -\frac{4}{3}, -\frac{14}{3}\right)$

3.
$$\begin{array}{rcrcrcr} -2x & + & 3y & - & z & = & -1 \\ x & + & 2y & - & 2z & = & 4 \\ -4x & + & 13y & - & 7z & = & 0 \end{array},$$
solution: no solution.

4.
$$\begin{array}{rcrcrcr} 4x & - & 2y & - & 3z & = & 2 \\ 2x & + & y & - & 5z & = & 1 \\ -2x & + & 7y & - & 9z & = & 1 \end{array},$$
solution: infinitely many solutions.

- Find the equation of the quadratic function that goes through the given points.

  1. $(1, 1), (2, 6), (-1, 9)$, Solution: $y = 3x^2 - 4x + 2$
  2. $(1, 2), (3, 8), (4, 11)$, (Solution will be a straight line.)

- Two liters of a 35% alcohol solution are to be made by mixing 60%, 40% and 15% solutions. The amount used of the 40% solution is supposed to be 3 times the amount used of the 60% solution. How much do we need to use of each solution?

  3 equations: total amount, amount of alcohol and proportion between the 60% and the 40%.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 0.6x & + & 0.4y & + & 0.15z & = & 0.7 \\ -3x & + & y & & & = & 0 \end{array},$$
solution: $\left(\frac{1}{3}, 1, \frac{2}{3}\right)$

Reading Quiz.

Notable Homework Problems.

- Project **??** can be assigned after this section.

Section 5.1. Sequences of Numbers

Suggested Time. $\frac{1}{2}$ class period

Lecture/Presentation.

- Motivation for engineering/science class. Measured data is always discrete and thus can be made into a sequence,

  Motivation for a CS-heavy class. All computer data is discrete and sequential. Connect to one-dimensional arrays, for example

  `http://physinfo.ulb.ac.be/cit_courseware/datas/data3.htm`.

- If the section is covered after the partial differential equations project PDE.I is discussed, sequences can be motivated through the sequences of coefficients defined there. Series can be motivated through the approximation of the initial condition with combinations of sine functions.

- If this module is covered in a first calculus class, one should be aware that this section will force those students who had some calculus before to think outside the box they think is calculus.

- Define sequences, point out differences and similarities between sequences and functions.

  Similarities: Sequences are special kinds of functions, can be graphed just like that. Sometimes sequences are sampled values of a function defined for all t (measurements, plotting points).

  Differences: We can think of sequences as a sequence of measurements we obtain as time progresses. The last number considered is always the current value we have, with the earlier numbers receding further into the "past" and upcoming numbers something we hope to predict.

  Visualization as "moving points".

- Examples of sequences. Harmonic, geometric, $a_k = k^2 - 50k$.

- Could mention recursively defined sequences, they will be covered in Sections 5.5 and 5.6.

- Introduce the definitions of increasing and decreasing as a first property that we might be interested in (amounts of radioactive material or bacteria, balance of a loan or a bank account).

  Discuss which of the given examples are increasing, decreasing, neither.

  Analyze the logical structure of the definition of increasing sequences and find the negation to show which are not increasing or decreasing.

Group Work/Examples.

- Students find the negation of "increasing"

- Students find the negation of "bounded above"

- Find a_{10} for each of the following sequences

  - $a_k = k^2 - 50k$,

  - $a_k = 2k + 5$

- Determine if the sequence is increasing

  1. $\left\{k^2 + 14k - 10\right\}_{k=0}^{\infty}$
  2. $\left\{\dfrac{k-1}{k+1}\right\}_{k=1}^{\infty}$

- Find an upper bound of the sequence and prove it is an upper bound.

  1. $\left\{\dfrac{k-1}{k+1}\right\}_{k=1}^{\infty}$
  2. $\left\{\dfrac{3k+2}{2k-1}\right\}_{k=1}^{\infty}$

- Find the next term in the sequence

  1. $1, 4, 7, 10, 13, 16, \cdots$
  2. $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \cdots$ (expand every second term with 2; this one also leads to the idea of subsequences)

- Do all increasing sequences grow beyond all bounds?

Notable Homework Problems.

- Exercise 6e shows that too much simplification can actually hinder an investigation.
- Exercise 7 shows that predicting the next term of a numerical sequence is always based on the assumption that there is a *simple* underlying law.

Section 5.2. Limits of Sequences

Suggested Time. 1 class period.

Lecture/Presentation.

- Example of data that stabilizes.

  The PolyChlorideBiphenal (PCB) content in a lake is measured every month. The content (in $\frac{\mu g}{l}$) over the last 10 months has been measured as 7.3, 6.5, 6.1, 5.8, 5.6, 5.5, 5.45, 5.42, 5.41, 5.405.

  Questions to ask. "Is this lake getting completely clean?" "What happens with the PCB content?"

- Define the limit. Motivate the idea with the desire to understand long-term behavior.

  Point out that "large enough" and "close to" are not precise terms.

- Lead in to limit laws for sequences.

  For which sequences do we really know the limit at $\pm\infty$?

  – $\lim_{k\to\infty} \frac{1}{k^r} = 0$ for $r > 0$

  Plus we can work out how limits interact with sums, differences, products etc.

- Idea for limits of rational expressions at infinity: Find the highest occurring power and expand with its reciprocal.

- Compute the limits of some sequences given by algebraic expressions,

  1. $\lim_{k\to\infty} \frac{2k^2+5k+2}{7k^2-2k+3}$
  2. $\lim_{k\to\infty} k \sin\left(\frac{1}{k}\right)$ (if l'Hôpital's rule is available)
  3. $\lim_{k\to\infty} \frac{3k^4+2k+1}{4k^2+8k^4-2} + \frac{k^2}{2^k}$

- Foreshadow the use of sequences in the analysis and definition of limits of functions (Section 5.2.2).

- (If this module is covered after calculus of one variable) Demonstrate the use of l'Hospital's rule,

- Can motivate the exact definition of the limit with the question how fast we are approaching the limit (the value at which the measurement stabilizes).

Group Work/Examples.

- Compute the limit of

  1. $a_k = \frac{5k^2-3k^3+2}{4k+3+7k^3}$

2. $a_k = \dfrac{3^k}{5k^{10} + 3k^4 + 12}$
3. $\lim\limits_{k\to\infty} \dfrac{5k^4 - 2k^3 + 13}{2k^5 - k^3 + 2}$
4. $\lim\limits_{k\to\infty} \dfrac{13k^2 + 4k - 2}{2k^2 + k^3 - 12}$
5. $\lim\limits_{k\to\infty} \dfrac{3k^3 + 10k^4 - 9}{2k^3 + 4k^4 - 8}$
6. $\lim\limits_{k\to\infty} \dfrac{\sqrt{8k^6 - 4k^4 + 3}}{-k^3 + 2}$
7. $\lim\limits_{k\to\infty} \dfrac{5k - 2k^3 + 13}{\sqrt[3]{2k^9 - k^3 + 2}}$
8. $\lim\limits_{k\to\infty} \dfrac{\sqrt{5k^6 - 2k^3 + 13}}{2k^3 - k + 2}$
9. $\lim\limits_{k\to\infty} \dfrac{\sqrt[4]{7k^4 + 31k^3 - 8}}{\sqrt{2k^4 - k^3 + 2}}$
10. $\lim\limits_{k\to\infty} \sqrt{k+1} - \sqrt{k}$
11. $\lim\limits_{k\to\infty} \sqrt{k+1} - \sqrt{k}$
12. $\lim\limits_{k\to\infty} \sqrt{4k^4 - k + 7} - 2k^2$

- For the given function f and the given point x_0, compute the limits $\lim\limits_{k\to\infty} f\left(x_0 + \dfrac{1}{k}\right)$ and $\lim\limits_{k\to\infty} f\left(x_0 + \dfrac{1}{k}\right)$.

  1. $f(x) = \dfrac{1}{(x-2)^2}$ at $x_0 = 2$
  2. $f(x) = \dfrac{|x+1|}{x+1}$ at $x_0 = -1$

Notable Homework Problems.

- Exercise 3q is a simple computation, but not what students might expect. Expanding with $\dfrac{1}{k^6}$ puts a $\dfrac{1}{k^{12}}$ under the square root, not a $\dfrac{1}{k^3}$. It is interesting to observe if students put a $\dfrac{1}{k^3}$ under the root or not. This is an indicator if the student does mathematics or if the student looks for schematic patterns in problems.

  I have to admit that I "invented" this problem when I made out a test. Checking the test I thought myself that the problem is "wrong". Then I realized the above. I would love to put more problems like this one anywhere in the text. Let me know if you have any. There are probably a number of blind eyes in the curriculum that could be thus opened.

- Exercises 4 and 5 foreshadow left- and right-sided limits and derivatives.

Section 5.3. The Exact Definition of the Limit of a Sequence

Suggested Time. $1-1\frac{1}{2}$ classes.

This section is likely the first time in the curriculum that students do significant work with inequalities. To account for this fact, more time and patience may be appropriate.

On the other hand, the conceptually most important part is Theorem 5.4.7, which characterizes divergence. If time is short, one could forego $\varepsilon - K$ arguments in favor of more coverage on how to prove that a sequence diverges. Theorem 5.4.7 would then almost have the status of an axiom.

Lecture/Presentation.

- There are formal problems with the intuitive definition of the limit that led to "contradictory" findings. Could be demonstrated with the alternating geometric series $(-1)^k$, which many are tempted to say "averages out to 0".

- There are also practical issues. Often one knows what the limit will be (say in radioactive decay or in a numerical scheme that is proven to give the correct result), but one wants to know "how fast" the limit is reached.

  Which goes to zero faster $\frac{1}{k}$ or $\frac{1000}{k^2}$?

- Present the exact definition of the limit. In the intuitive definition, "large enough" and "close to" are not precise terms.

  - "Close to" depends on the context (near miss of an asteroid can be around 150,000mi, buildings that are close to each other are less than a mile apart, instruments in a car that are close to each other are at most a few inches apart).
  - "Large enough" also depends on the context (extragalactic objects that are large enough to see with a telescope are at least the size of a sun; with the naked eye, they need to be galaxies; a lithograph on a microchip that is deep enough to print circuits on it is by now measured in layers of hundreds of atoms or less).

  For a mathematical notion we want to be able to fit all context (to be context-independent). The exact definition allows us to do this.

  We have convergence if for *any* tolerance ε we decide to give (no matter how small, this is the precise meaning of "close to"), the sequence will eventually (for k exceeding a certain K, this is what "large enough" means) be closer to the limit than the tolerance value.

- Break down the exact definition line by line to show the scope of the different quantifications.

  The sequence $\{a_k\}_{k=s}^{\infty}$ is said to converge to the number L

  if and only if:

  For every $\varepsilon > 0$
  there is an integer $K \geq s$ such that
  for all integers $k \geq K$ we have
  $|a_k - L| < \varepsilon$.

- Show that the limit of $\dfrac{1}{k^2+1}$ is zero.
- Show that the limit of $\dfrac{3k+1}{2k-5}$ is $\dfrac{3}{2}$ using the definition (careful when to drop the absolute values). Use the proof to find values for K such that the sequence is within .1, .01, $\cdots$ of the limit for $k > K$.
- Show the power of the negation of the definition of convergence, which can be used to clarify when something does not converge. (This is challenging with the intuitive definition. Demonstrate with $(-1)^n$.)
- Show that the sequence 2^k grows beyond all bounds.
- Prove that the product of a bounded sequence with a sequence that goes to zero is converges to zero.

Group Work/Examples.

- Determine for what value of K the sequence $\dfrac{k^2-1}{k^2+4}$ is within 10^{-3} of its limit.
- Show that the sequence $\{\sin(\pi k)\}_{k=0}^{\infty}$ converges.
- Possible lead in to Section 5.4. Group question: "Give an example of a divergent sequence." (May need to hint that sequences that go beyond all bounds are also divergent.)

  Analyze examples given by groups. Note that all examples either oscillate or go to infinity. The question we want to answer is if there is another "failure mode" for convergence.

  Analysis of failure modes is common in engineering and in programming. If we understand how a device breaks or how a program locks up, we have taken a step towards fixing the device/program.
- If a sequence is always close to zero, except that every 1000^{th} term is 1, does this sequence converge to zero?

Reading Quiz.

- Among the following statements about the precise definition of the limit, check all that apply. The precise definition of the limit ...

  ☐ ... Is useful purely for formal reasons that don't affect anything but mathematics.

  ☐ .. Can be useful any time one has to consider not just convergence of a sequence but also the "speed" of the convergence.

  ☐ ... Is not useful at all.

  ☐ ... Gives us a different notion of convergence than the intuitive definition.
- A sequence $\{a_k\}_{k=s}^{\infty}$ converges if and only if

- ☐ There is a number L, called the limit, such that for each $\varepsilon > 0$ there is a natural number K such that for all $k \geq K$ we have $|a_k - L| < \varepsilon$.
- ☐ For every number L, called the limit, there is an $\varepsilon > 0$ and a natural number K such that $|a_K - L| < \varepsilon$.
- ☐ Terms of the sequence get really really close to some number.
- ☐ None of the above.

- A sequence $\{a_k\}_{k=s}^{\infty}$ diverges if and only if
  - ☐ There are two numbers L and M such that for each $\varepsilon > 0$ there is a natural number K such that for all $k \geq K$ we have $|a_k - L| < \varepsilon$ and $|a_k - L| < \varepsilon$.
  - ☐ There is a subsequence with unbounded absolute values or there are two subsequences that converge to different limits.
  - ☐ The sequence is unbounded.
  - ☐ The sequence has two subsequences that have different limits.

Notable Homework Problems.

Section 5.4. Subsequences – A Better Characterization of Divergence

Suggested Time. $1 - 1\frac{1}{2}$ classes.

Lecture/Presentation.

- Possible intro: Use the sequence $a_k = \begin{cases} \frac{2^{\frac{k+1}{2}}-1}{2^{\frac{k+1}{2}}}; & \text{for } k \text{ odd,} \\ \frac{1}{2^{\frac{k}{2}}}; & \text{for } k \text{ even,} \end{cases}$ to give a numerical intro. DON'T put this expression on the board. Put up the first few terms $1, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{7}{8}, \frac{1}{8}$, etc. and ask students what patterns they see. Give first few terms, ask for next few terms and lead into the fact this one is "made up of two sequences".
- Define subsequences. Note the analogy to composition.
- Explain the characterization of divergence (Theorem 5.4.7).

  Students could be tied in by letting them find negations of statements such as the statement of convergence to L.

  Motivate that if the sequence does not converge to L there must be a subsequence that stays away from L (Lemma **??**). Then show how this breaks up into the characterization.
- Give examples how subsequences can be chosen to show that a sequence is divergent. (The sequences below are also investigated in group work.)
  1. $\left\{(-1)^k\right\}_{k=1}^{\infty}$
  2. $\left\{\left(1+(-1)^k\right)2^k\right\}_{k=1}^{\infty}$

Group Work/Examples.

- For the sequence $\{a_k\}_{k=1}^{\infty} = \left\{\left(1+(-1)^k\right)2^k\right\}_{k=1}^{\infty}$ find the subsequences $\{a_{2m}\}_{m=1}^{\infty}$ and $\{a_{2m+1}\}_{m=1}^{\infty}$ (simplify as much as possible).
- For the sequence $\{(-1)^k\}_{k=1}^{\infty}$ find a subsequence that converges to 1.
- For the sequence $\left\{\sin\left(\frac{\pi k}{24}\right)\right\}_{k=1}^{\infty}$ find a subsequence that converges to $\frac{1}{2}$.
- Show that the sequence $\left\{\sin\left(\frac{\pi}{2}k\right)\right\}_{k=0}^{\infty}$ does not converge.
- Activity on page ACT-14.

Notable Homework Problems.

- Exercise 3 is very simple, but turns out to be quite challenging for some students.
- Exercises 10 and 11 are part of the "logic strand" through the exercises. The idea is to make students reformulate and reverse theorems to gain more exposure to the theorems.

Section 5.5. Recursively Defined Sequences

Suggested Time. $\leq \frac{1}{2}$ class.

Lecture/Presentation.

- Define recursive sequences.

  Motivation for computing oriented students. Recursion is an elegant, if inefficient, way to program an algorithm or a computation. Several exercises are set up to exhibit the inefficiency of recursion.

  At the same time a recursive formulation is often the first step towards a solution. In a separate step the recursion is then often resolved into an iteration.

- Examples of recursive sequences. Interest, Fibonacci.

- Define the factorial function, (possibly as a first example)

- Demonstrate the resolution of a recursion, for example with $a_{k+1} = \frac{a_k}{2}, \ a_0 = 1$.

  Compare computing speeds (with CAS if available and if the CAS is slow enough)

  (Also show that this sequence is decreasing.)

- Discuss the elegance as well as the inefficiency of recursion,

- If the section is covered close to the series solution of differential equations, heavily stress the resolution of recursion as an upcoming needed technique.

Group Work/Examples.

- Find a_5 for

  1. $a_k = \frac{3a_{k-1} + 2}{4}, a_1 = 6$
  2. $a_k = 3a_{k-1} + 2a_{k-2}, a_0 = 0, a_1 = 1$
  3. $a_k = 2^k$ and $a_0 = 1, a_{k+1} = 2a_k$. Which computation was faster?

  Determine which is increasing or decreasing.

- Resolve the recursion $a_{k+1} = \frac{10a_k}{k}$. Are these numbers "getting close" to some number for large indices?

- If capabilities and facilities allow it, program a recursive sequence, evaluate it for large indices.

Notable Homework Problems.

- Exercise 7 shows how a method similar to what is done in Section LDλ.A for differential equations can be used to resolve some recursions.

- Exercise 10 is another well-known brainteaser.

Section 5.6. Limits of Recursive Sequences

Suggested Time. ≤ 1 class

Lecture/Presentation.

- When determining limits of recursively defined sequences, it is always necessary to prove that the limit exists.
- To illustrate Theorem 5.6.2 (Monotonic Sequence Theorem), note that 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, $\cdots$ is an increasing sequence that is bounded above by $\sqrt{2}$ (or a larger number). Yet in the rational numbers it does not have a limit.
- Compute the limits of some sequences given by recursive definition, prove that the limits exist first,
  1. $a_{k+1} = 10 - \dfrac{20}{a_k}, a_0 = 7$
  2. Could refer to the sequence $a_{k+1} = \dfrac{a_k}{2} + \dfrac{1}{a_k}$, $a_1 = 1$ of Exercise 3 as an example of an important sequence for which it is hard to show it is increasing and bounded.

     The sequence is generated by using Newton's method on x^2-2. If Newton's method has been covered, this is another good cross-connection. If not, this can be used to foreshadow iterative techniques.
- Show an example of a recursively defined sequence that diverges, yet the limit computation under the assumption there is a limit gives a numerical answer. $a_{k+1} = 2a_k^2$, $a_1 = 1$.
- Show that the sequence above can be said to have limit ∞ if one chooses to include this type of analysis.

Group Work/Examples.

- Why is the Monotonic Sequence Theorem an axiom that distinguishes the rational numbers from the real numbers?
- Compute the limit of
  - $a_{k+1} = \sqrt{a_k}, a_1 = 5$

Notable Homework Problems.

Section 6.1. Rational Functions

Suggested Time. $\frac{1}{2}$ class period. To motivate limits, part of Section 9.1 could also be covered here.

Lecture/Presentation.

- Define rational functions
- Give examples of the different types of behavior that rational functions can exhibit at $\pm\infty$ as well as at places where the denominator is zero.

Group Work/Examples.

- For each function, describe the behavior for large values of x. In what ways do the behaviors differ from each other or from that of polynomials?

  1. $f(x) = \dfrac{-x^3 + 2x^2 + 15}{2x^3 - x^2 + 1}$
  2. $f(x) = \dfrac{2x^3 + 4x^2 + 2}{2x^2 + 3x}$

  Can also connect this with the formal analogy to limits of sequences.

- For each function, describe the behavior near $x = 1$. In what ways do the behaviors differ from each other or from that of polynomials?

  1. $f(x) = \dfrac{x^2 - 1}{x - 1}$
  2. $f(x) = \dfrac{2x^2 + 1}{x - 1}$

Reading Quiz.

Notable Homework Problems.

Section 6.2. Limits of Functions at Infinity

Suggested Time. 1 class period.

Lecture/Presentation.

- How do we measure the long-term behavior of a quantity? Graphical example: Number of words a typist can type as experience increases (learning curve leveling off at 70, this is more like a sequence).

  Blood sugar level after a meal with a sugary dessert. (This is a continuous quantity. You have a blood sugar level at any given time.) [The normal amount of sugar in the blood ranges from about 70 mg/dL to about 120 mg/dL in people who don't have diabetes. Pick a value in this range as the asymptote. Could also give an example of a graph that does not level off and ask what the problem is.]

  The underlying parameter now is continuous (though we could discretize, since the measurements are discrete).

- We say "as x approaches ∞, $f(x)$ approaches 70" in the above example.

- ∞ is not a number. It is a symbol to tell us that a quantity increases without bound.

- Informal definition of the limit at infinity.

  Let f be a function defined on an interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ is arbitrarily close to L for all sufficiently large values of x.

- Present the formal definition of the limit at infinity and explain the connection.

- Graphical ramification.

  If $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$ then $y = b$ is called a horizontal asymptote of f. (sketch it in the picture)

- Asymptotes as $x \to -\infty$: looking at the distant past, ex: picture, $f(x) = \dfrac{1}{x}$, size of the universe.

- Lead in to limit laws for functions.

  For which functions do we really know the limit at $\pm\infty$?

- 1. $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$
  2. $\displaystyle\lim_{x\to-\infty} \frac{1}{x^r} = 0$ for $r > 0$ if it is defined (ask for an r for which it isn't)

- Other functions are made up of these pieces, so how do we get access to them?

- Examples: Find the horizontal asymptotes (if they exist) of

  1. $f(x) = 4 + \dfrac{3}{x}$ as $x \to \infty$,
  2. $f(x) = \dfrac{5 - \frac{2}{x^2}}{2 + \frac{7}{x^5}}$ as $x \to -\infty$,

3. $f(x) = \dfrac{5x^3 - 2x + 4}{3x^3 + 2x^2 - 8}$ as $x \to \infty$,
4. $f(x) = \dfrac{\sqrt{3x^2 + 2}}{x - 5}$ as $x \to \infty$ and as $x \to -\infty$,
5. $f(x) = \sqrt{x+1} - \sqrt{x}$ as $x \to \infty$,

- Idea remains the same as for sequences. Find the highest occurring power and expand with its reciprocal.
- Prove the characterization of what it means that a function does not have a limit at ∞ (Theorem 6.3.10).

  This characterization is a recurring theme (cf. Theorems 6.5.10 and 6.6.14). The proof can be shifted more and more to the student, which is suggested in this guide. Laying a solid foundation with this discussion will make this task easier.
- Prove that $\cos(x)$ has no limit as $x \to \infty$.
- Define an infinite limit
- Define asymptotic equality and compute an asymptotically equal expression for one of the rational function examples that had an infinite limit.
- Refer back to the rational functions intro in Section 6.1 and explain that we have behavior at $\pm\infty$ under control now.

Group Work/Examples.

- Find examples of

  1. A real life process with a horizontal asymptote (one example zero, one example not zero),

     ex: radioactive decay, heat put out by a heat source that was just turned on,
  2. A function $f(x)$ with a horizontal asymptote (one example zero, one example not zero),

     ex: $f(x) = \dfrac{1}{x}\sin(\pi x)$, $f(x) = 1 + \dfrac{1}{x}$
  3. The graph of a function with a horizontal asymptote (one example zero, one example not zero),

- Find the horizontal asymptotes (if they exist)

  1. $f(x) = \dfrac{1}{1+x^2}$,
  2. $f(x) = \dfrac{3x^2 - 5x + 2}{-2x^2 + x + 1}$,
  3. $f(x) = \dfrac{4x^3 - x^2 + 2x + 5}{2x^2 - 1}$

- Compute the limit if it exists.

  1. $\lim\limits_{x\to\infty} \tan^{-1}(x)$,

2. $\lim_{x\to-\infty} \tan^{-1}(x)$,
3. $\lim_{x\to-\infty} x^2 - 200x + 1$,

Reading Quiz.

Notable Homework Problems.

- The answer to Exercise 8 is "no" and Exercise 15a is an example to show this.
- Exercise 11 presents the "big-Oh" notation used to describe growth behavior and algorithm performance.
- Exercises 12 and **??** are part of the logic strand through the text. The idea is to acquaint students with theorems by reformulating and reversing them.

Section 6.3. Theoretical Background: The Precise Definition of the Limit at $\pm\infty$

Suggested Time. 1 class period.

Lecture/Presentation.

- This is a new section, created to separate the deeper aspects from the intuitive definition, so there are no manual entries yet.

Group Work/Examples.

Reading Quiz.

Notable Homework Problems.

Section 6.4. Limits of Functions at a Point

Suggested Time. 1.5 class periods. First show informal definition, limit laws and graphical representation. Then talk about showing divergence, formal definition of the limit. Can weave in the start of Section 9.1 for another motivation why we would want to "divide by zero".

Lecture/Presentation.

- We say the limit of $f(x)$ as x approaches a equals the number L if $f(x)$ is close to L for all values x close to but not equal to a.

  Notation: $\lim_{x\to a} f(x) = L$ or $f(x) \to L$ as $x \to a$

- Show examples of limits that exist and of limits that do not exist (say $\sin\left(\frac{1}{x}\right)$ at $x = 0$ or $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$)

- Make the idea precise. Definition of the limit at a point via limits of sequences.

- Standard test for divergence. Send one sequence towards the point from the left and another from the right, usually $\left\{p + \frac{1}{k}\right\}_{k=1}^{\infty}$ and $\left\{p - \frac{1}{k}\right\}_{k=1}^{\infty}$. This does not cover all cases, but it is a start.

- Graphical examples where the limit exists Also need graphical examples where the limit does not exist.

- Limits of polynomials are trivial

- $\lim_{x\to -3} \frac{2x^2+5x-3}{x+3}$ (show first with GC) factor and solve

- Show the traps in the numerical approach by graphing the function $f(x) = \frac{\sqrt{x+16}-4}{x}$, in $[-10^{-6}, 10^{-6}] \times [.124999, .125001]$, which shows how rounding errors again enter the picture.

- Note that any approach via tables of values only shows that convergence *might* happen along one particular approach.

- Do the above problem algebraically on the board,

- Motivate the limit laws as an abbreviation of the computation process

- Present the limit laws

  If a, c, r are constants and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, Then

$$
\begin{aligned}
\lim_{x\to a}(f+g)(x) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x), \\
\lim_{x\to a}(f-g)(x) &= \lim_{x\to a} f(x) - \lim_{x\to a} g(x), \\
\lim_{x\to a}(f\cdot g)(x) &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x), \\
\lim_{x\to a}\left(\frac{f}{g}\right)(x) &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ if } \lim_{x\to a} g(x) \neq 0 \\
\lim_{x\to a} f^r(x) &= \left(\lim_{x\to a} f(x)\right)^r, \text{ if the powers exist.}
\end{aligned}
$$

$$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- Example: $\lim_{x \to 2} \sqrt{8 - x^2} = \sqrt{\lim_{x \to 2}(8 - x^2)} = \sqrt{8 - 4} = 2$
- Examples with cancellations (Recall stuff on rational functions. Justify with theorem that if $g(x) = f(x)$ near a, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.)

  1. $\lim_{x \to 3} \dfrac{2x^2 - 5x - 3}{x - 3}$
  2. $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

- Compute some limits using that $\lim_{x \to p} f(x) = \lim_{h \to 0} f(p + h)$

Group Work/Examples.

- Activities on pages ACT-16 and ACT-17.
- Students formulate what it means that the limit does not exist at a point.

  This is Theorem 6.5.10. Having seen a presentation of Theorem 6.3.10, students should be able to come up with the formulation after some prompting.

- Show that $\sin\left(\dfrac{1}{x}\right)$ diverges at $x = 0$
- $\lim_{x \to 0} \dfrac{\sqrt{x + 16} - 4}{x}$, suggest a numerical approach ($x = .1, .01, .001, \cdots$) or an algebraic aproach with $h_n \to 0$.
- $\lim_{x \to 0} \dfrac{\sin(x)}{x}$ (graphically)
- *If it comes up.* The effect of working in degrees in the above problem:

$$\lim_{x \to 0} \frac{\sin_\circ(x^\circ)}{x^\circ} = \lim_{x \to 0} \frac{\sin_\circ\left(x(\text{rad}) * \frac{180^\circ}{\pi}\right)}{x(\text{rad}) * \frac{180^\circ}{\pi}} = \lim_{x \to 0} \frac{\sin_{\text{rad}}\left(x(\text{rad})\right)}{x(\text{rad}) * \frac{180^\circ}{\pi}} = \frac{\pi}{180^\circ}$$

- $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x}$ (graphically)
- $\lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$
- For $f(x) = \begin{cases} 0; & \text{if } x \leq 0, \\ 1; & \text{if } x > 0. \end{cases}$ find $\lim_{x \to 0} f(x)$ or show that it does not exist.. (Leads into left and right-sided limit.)
- Suppose $\lim_{x \to 1} f(x) = 2$ and $\lim_{x \to 1} g(x) = 3$. Compute

  1. $\lim_{x \to 1} (3f(x) + g(x))$
  2. $\lim_{x \to 1} \left(\dfrac{f(x)}{g(x)}\right)$

3. $\lim_{x\to 1}(f(x)g(x))$

- Application of the theorem "bounded times zero goes to zero": $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$.
- **Maybe at end?** Is there an a such that $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exists? If so, what is it?
- Worksheet on next page.
- Pyramid workout **??**.

Reading Quiz.

Notable Homework Problems.

- Some Exercises, such as for example 5l, 5j, and 5k, are much easier to solve using $\lim_{h\to 0} f(p+h)$.
- Exercise 6 foreshadows some of the computations needed to find the derivatives of trigonometric functions
- Exercises 9, 7 and **??** are part of the logic strand through the text. The idea is to acquaint students with theorems by reformulating and reversing them.

Section 6.5. Theoretical Background: The Precise Definition of the Limit at a Point

Suggested Time. 1 class period.

Lecture/Presentation.

- This is a new section, created to separate the deeper aspects from the intuitive definition, so there are no manual entries yet.

Group Work/Examples.

Reading Quiz.

Notable Homework Problems.

Section 6.6. One-Sided Limits and Vertical Asymptotes

Suggested Time. 1class period.

Lecture/Presentation.

- We say $\lim_{x \uparrow a} f(x) = L$ if $f(x)$ is close to L for all x sufficiently close to a, but less than a.

  Then let students formulate the exact definition using sequences.

- Graphical example of left and right sided limit

- Sketch a function that has certain (left or right) limits at certain points. (Students find limits.)

- Examples with greatest integer function: Find limit or, if the limit does not exist, left and right sided limits at 1.5, 2, -2, $-\pi$

- Theorem: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

- Example: $\lim_{x \to 2} |x - 2| = 0$ But recall: no tangent at $x = 2$.

- Lead in to vertical asymptotes: What happens with $\dfrac{1}{x^2}$ as $x \to 0$?

- If $|f(x)| \to \infty$ as $x \to a$ from the left or right, then $x = a$ is called a vertical asymptote.

  Can introduce the definition of an infinite limit here.

- Often this happens when a denominator gets small and the corresponding numerator does not.

- Discuss three examples symbolically and graphically to show behavior near vertical asymptotes (I do first, they do others in groups.)

  1. $f(x) = \dfrac{1}{x - 2}$,
  2. $f(x) = \dfrac{1}{(x + 3)^2}$,
  3. $f(x) = -\dfrac{1}{(x - 1)^4}$,
  4. $f(x) = \dfrac{1}{1 - x}$,

- A zero in the denominator is a hint at a vertical asymptote, but not a guarantee, cf. $f(x) = \dfrac{x - 2}{2 - x}$

  "If the denominator is zero, then you should check for a vertical asymptote." good advice, but not a logical statement.

  "If the denominator is zero, then there is a vertical asymptote." FALSE logical statement.

- Real life examples of vertical asymptotes: Resonance phenomena.

- If f is a function defined near a, then we say $\lim_{x \to a} f(x) = \infty$ if and only if $f(x)$ exceeds any given boundary, for all numbers x sufficiently near a.

- Left and right limits and limits that turn out to be $-\infty$ are defined similarly.
- Examples

  1. $\lim_{x\to 0} \frac{1}{x^2}$
  2. $\lim_{x\to -2^-} \frac{1}{x+2}$
  3. $\lim_{x\to 1^+} \frac{x-3}{(x-1)(x+2)}$
  4. $\lim_{x\to 2^+} \frac{x^2-2x+3}{x^2-5x+6}$
  5. $\lim_{x\to 0^+} \ln(x)$
  6. $\lim_{x\to (\frac{\pi}{2})^+} \frac{x^2}{(x-2)^2\cos(x)}$
  7. $\lim_{x\to 0} \ln(x)$
  8. $\lim_{x\to 0} \ln|x|$
  9. $\lim_{x\to (\frac{\pi}{2})^+} \tan(x)$

Group Work/Examples.

- Students formulate the definition of the left-sided limit via sequences
- Students formulate what it means that the left-sided limit does not exist.
- $\lim_{x\downarrow 0} x2^{\frac{1}{x}}$ or $\lim_{x\downarrow 0} \frac{1}{1+2^{\frac{1}{x}}}$ (would need to be done graphically)
- $\lim_{x\uparrow 0} x2^{\frac{1}{x}}$ or $\lim_{x\uparrow 0} \frac{1}{1+2^{\frac{1}{x}}}$ (would need to be done graphically)
- $\lim_{x\to 0} x2^{\frac{1}{x}}$ or $\lim_{x\to 0} \frac{1}{1+2^{\frac{1}{x}}}$ (would need to be done graphically)
- Group work: Find the vertical asymptotes (if they exist) (also look at them graphically)

  1. $f(x) = \frac{1}{(x-1)(x+2)}$,
  2. $f(x) = \frac{x^2-1}{x-1}$,
  3. $f(x) = \frac{1}{.001+(x-3)^2}$, (calculator can deceive you)

Reading Quiz.

Notable Homework Problems.

- Exercise 11 is part of the logic strand through the text. The idea is to acquaint students with theorems by reformulating them.

Section 6.7. Continuity

Suggested Time. 1 class period

Lecture/Presentation.

- Functions for which limits are easy to evaluate (recall polynomials): Continuous Functions.
- A function is continuous at a number a if and only if
  1. $f(a)$ is defined,
  2. $\lim_{x \to a} f(x)$ exists,
  3. $\lim_{x \to a} f(x) = f(a)$
- Graphical example: where is a given function discontinuous? Highlight jump discontinuities, removable discontinuities, infinite discontinuities and discontinuities by oscillation.

  Give the reasons why they are discontinuities in terms of the definition of continuity.

  Possibly make a table: place, justification with definition, name of disc.
- Show theorem that these are all types of discontinuities (Theorem 6.7.11).
- Symbolical example: Where is $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 4x + 3}$ discontinuous?
- Left and right continuity
- Examples for left and right continuity:
  1. $f(x) = \lfloor x \rfloor$
  2. $g(x) = \dfrac{|2x+3|}{2x+3}$ (puzzle the definition of $|x|$ apart)
- Turning the local concept of continuity at a point into a global concept: Continuity on an interval.
- A function f is continuous on the interval I if and only if for all points a in I, f is continuous at a,
- Graphical interpretation: continuous functions can be drawn in one pencil-stroke (without the pencil ever lifting from the paper). Connection with the formal laws: At every point a in the interval
  1. $f(a)$ is defined, "the pencil has a place to go to"
  2. $\lim_{x \to a} f(x)$ exists, "the approach towards a is orderly"
  3. $\lim_{x \to a} f(x) = f(a)$, "the approach meets the target point".
- Graphical examples: Draw several functions that are or are not continuous, let them find out which is which. Throw in the graph of a circle. (Not a function. Shows that if one gets too focused on one property, one might forget others.)

- Quickly review the graphical example and the symbolic example from a graphical point-of-view.
- Prove that $f(x) = \dfrac{4x+1}{x-2}$ is continuous in the interval $[-1, 1]$, (using limit laws)
- If f, g are continuous, then so are $f+g$, $f-g$, $f \cdot g$, and $\dfrac{f}{g}$ (for all points where $g(a) \neq 0$)
- So again we have ways to build more complex continuous functions from simpler ones. What types of functions can we start with?
- Functions that are continuous on their respective domains:

  1. polynomials,
  2. rational functions,
  3. root functions,
  4. trigonometric functions,
  5. inverse trigonometric functions, (may want to quickly review these)
  6. (exponential functions, later)
  7. (logarithmic functions, later).

- Where is $f(x) = \dfrac{x + \arcsin(x)}{x^2 - 4}$ continuous?
- If f is continuous at $g(a)$, and g is continuous at a, then $f \circ g(x) = f(g(x))$ is continuous at a.

Group Work/Examples.

- Where are the following functions discontinuous?

  1. $f(x) = \lfloor x \rfloor$
  2. $g(x) = \begin{cases} \frac{x^2+6x+5}{x+1}; & \text{if } x \neq -1, \\ 5; & \text{if } x = -1. \end{cases}$
  3. $h(x) = \begin{cases} \frac{\sin(x)}{x}; & \text{if } x \neq 0, \\ 1; & \text{if } x = 0. \end{cases}$
  4. $j(x) = \begin{cases} \frac{1-\cos(x)}{x}; & \text{if } x \neq 0, \\ 1; & \text{if } x = 0. \end{cases}$

- Find a value of c such that $f(x) = \begin{cases} cx - 4; & \text{for } x \leq 1, \\ x - c; & \text{for } x > 1. \end{cases}$ is continuous (at 1).
- Where is $f(x) = \sqrt{1 - x^2}$ continuous?

Reading Quiz.

Notable Homework Problems.

Section 6.8. Consequences of Continuity

Suggested Time. 1 class period

Lecture/Presentation.

- Explain the idea behind working with a property rather than working with examples. Knowledge about a property takes care of an infinite number of examples all at once.
- The intermediate value theorem. Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there is a number c in $[a, b]$ such that $f(c) = N$.
- Explain the logic of the IVT.

  Show that if the hypothesis is not satisfied, the conclusion will not hold, even for "nice" functions like $f(x) = \dfrac{1}{1 - 2^{\frac{1}{x}}}$, which has no zero between -1 and 1.
- Demonstrate the bisection method with some continuous function.
- Show how to determine where a given function is positive or negative.

  Note that a function can only change signs where it is zero or where it is discontinuous.

Group Work/Examples.

- Find where the function is ≥ 0.

  1. $f(x) = \dfrac{x^2 - 4}{x^2 + 3x - 18}$
- Determine the domain of the function $f(x) = \sqrt{\dfrac{3x + 1}{x^2 - 3x + 2}}$
- Use the bisection method to find, for each zero of the function, an interval of length $< 10^{-2}$ that contains the zero.

  1. $f(x) = x^3 - 4x + 2$
- Is there a solution to the equation $\cos(x) = x$? If so, use the bisection method to find an interval of length $\leq 1/100$ that contains it.
- Find the absolute maximum of the function on the interval or explain why Theorem 6.8.11 does not apply.

  1. $f(x) = 2 + |x - 1|$ on $[0, 2]$
  2. $f(x) = \dfrac{1}{x}$ on $(0, 1]$

Reading Quiz.

Notable Homework Problems.

- Exercises 12 and 13 are part of the logic strand through the text. The idea is to acquaint students with theorems by reformulating and reversing them.

Section 6.9. Sketching Rational Functions Using Algebra and Limits Only (Optional)

Suggested Time. $\frac{1}{2}$ class period. Graphing is most appropriately done once we have differential calculus available. Early coverage will be incomplete, but it can help set the stage.

Lecture/Presentation.

- Explain the procedure on page 241.

Group Work/Examples.

- Sketch the graph of the function using the procedure on page 241.

  1. $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$
  2. $f(x) = \dfrac{(x-3)(x+2)}{(x-4)(x+1)(x+2)} = \dfrac{x^2 - x - 6}{x^3 - x^2 - 10x - 8}$
     (give in factored form if it is to be used as an example)

Reading Quiz.

Notable Homework Problems.

Section 6.10. Continuous Extensions

Suggested Time. $\frac{1}{2}$ class period. This section has recently been shrunk and right now it just presents some deep background info.

Lecture/Presentation.

- Talk about the fact that $2^x = 3$ has no rational solution. (Proof by contradiction.)
- "Philosophical issue" (?) or neat analogy. I may be a bit tired today, but the following might be worth considering. Just as most of the universe is dark matter which we cannot see, most numbers are irrational and yet we rarely see them. However the effects of dark matter are measurable just as the irrational numbers have profound significance in mathematics. This could be parlayed into another motivation why we need to consider of irrational exponents.
- Explain how powers with irrational exponents are defined by continuous extension of the power function for rational numbers.
- Introduce the natural exponential function as the most frequently used exponential function because it has a nice derivative (whatever "derivative" might mean).
- Note that $\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^{\frac{1}{k}} = e^x$.

Group Work/Examples.

- The whole section is quite theoretical. Experimental verifications of claimed limits should be very helpful here. Unless a class is very strong in theory, presentation or practice of proofs will probably not be needed.

Reading Quiz.

Notable Homework Problems.

Section 7.1. Exponential Functions

Suggested Time. 1 class period

Lecture/Presentation.

- Say you were offered a job to the following conditions. The guaranteed employment period is two years (if your employer terminates the employment before then, you will be paid the full salary for the two year period immediately).

  Salary structure. The first month's salary is one dollar. The salary will be doubled every month.

  Would you take this job?

- Make a table and sketch $f(x) = 2^x$. (or function related to the lead in exercise.)

- Allude to the definition of exponential functions via continuous extension of the power function with rational exponents.

- The exponential function with base $a > 0$ is denoted by $f(x) = a^x$ where x is any real number.

- Question for audience: Do all exponential functions increase? What would be the range of an exponential function?

- Sketch graphs of exponential functions for $a > 1$ (MAIN FOCUS). Give properties as in theorems listed.

  Briefly discuss $0 < a < 1$.

- Could show $\lim_{x\to\infty} e^{-x} = 0$ via reflection rule. This sets the stage for a possible proof of Theorem 7.1.12 by the student.

- Examples of limits at infinity involving exponentials.

  1. $\lim_{x\to\infty} e^{-x}$,
  2. $\lim_{x\to\infty} e^{-\frac{1}{10}x}\cos(3x)$,
  3. $\lim_{x\to\infty} e^{x}$,
  4. $\lim_{x\to-\infty} e^{-x}$,

Group Work/Examples.

- Graph the function $f(x) = 2 + 3^{x-1}$ using shift rules. Check with GC.

- Fill in the table of rules for exponential functions

$$
\begin{aligned}
a^0 &= 1 \\
a^{-x} &= \frac{1}{a^x} = \left(\frac{1}{a}\right)^x \\
a^x a^y &= a^{x+y}
\end{aligned}
$$

$$\begin{aligned}
\frac{a^x}{a^y} &= a^{x-y} = \frac{1}{a^{y-x}} \\
(a^x)^y &= a^{xy} \\
(ab)^x &= a^x b^x \\
\left(\frac{a}{b}\right)^x &= \frac{a^x}{b^x}
\end{aligned}$$

- Write the following as one exponential function a^x.

  1. $f(t) = 4^{2t}\left(4^{-3t}\right)^2$
  2. $g(t) = \dfrac{2^{t^2}}{8}$
  3. $g(t) = \dfrac{3^t}{9^{4t}}$
  4. $g(t) = \sqrt[3]{7^t 7^{t^3}}$

  Show that the two functions are equal or find a t at which they do not agree.

  1. $\dfrac{2^t}{5^t} = .4^t$
  2. $3^t + 4^t = 5^t$
  3. $2^{t^3} = 2^{3t}$
  4. $\left(2^t\right)^3 = 2^{3t}$

  Recall the logic stuff we did. Each equation would have to be true for all t.

- Aside from lengthy computations, Project BVP.G could be done at this stage.

Reading Quiz.

Notable Homework Problems.

- Exercise 9 is an alternative proof of Theorem 7.1.14.

Section 7.2. Logarithmic Functions

Suggested Time. 1 class period

Lecture/Presentation.

- $5\mu g$ of Uranium 228 obey the decay equation $m(t) = 5e^{\frac{-.076t}{\min}}\mu g$. How long does it take until only $2.5\mu g$ are remaining? (about 9.1min, which is the half life of this substance)

  1. Done graphically: need to choose the right window, get estimate
  2. ask: how can we get more precision?
  3. motivation for ln: get the result faster solve the equation using ln

- logarithms to the base a ($a > 0$, $a \neq 1$)

  - $\log_a(x)$ is "that number to which a has to be raised to obtain x"
  - $y = \log_a(x)$ iff $a^y = x$
  - domain of the logarithm: $(0, \infty)$

- Obtain graph of $\ln(x)$ by reflecting e^x over the diagonal.

- Give graphs of $\log_a(x)$ for $a > 1$ (MAIN FOCUS). Do $0 < a < 1$ later or just mention them.

- $y = \ln(x + 2)$: horizontal shift $y = \ln(x) + 2$: vertical shift

- Simplify $2^{\log_2(15x)}$, $\log_3\left(3^{4x+1}\right)$.

- State
  $b^{\log_b(u)} = u$ for $u > 0$,
  $\log_b(b^u) = u$ for all u.

- Properties of logarithms (explain verbally, compare with corresponding exponential property, give proof)

  1. $\log_a a = 1$,
  2. $\log_a 1 = 0$,
  3. $\log_a(xy) = \log_a(x) + \log_a(y)$
     Proof. $\log_a(xy) = \log_a\left(a^{\log_a(x)}a^{\log_a(y)}\right) = \log_a\left(a^{\log_a(x)+\log_a(y)}\right) = \log_a(x) + \log_a(y)$
  4. $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$
     Proof. Similar to the above.
  5. $\log_a(x^r) = r\log_a(x)$
     Proof. $\log_a\left(x^r\right) = \log_a\left(\left(a^{\log_a(x)}\right)^r\right) = \log_a\left(a^{r\log_a(x)}\right) = r\log_a(x)$

- Some exercises on combining and expanding logarithms.

- To show how logarithms are used to reduce the size of "large" quantities (compressing scales):

  A Richter no. R for an earthquake is given by

  $$R = \log_{10} \frac{I}{I_0},$$

  where I is the intensity of the quake and I_0 is the intensity of a reference quake (or: smallest shift measurable?)

  Question: A 1990 quake in Iran was a 7.7 on the Richter scale. A 1989 quake in San Francisco was a 6.9 on the Richter scale. How many more times intense was the quake in Iran?

  Group exercise with hints?? Tell them to solve for the intensities and then divide?

- Briefly present change-of-base formula as a consequence of solving $a^x = b$ for x.

  $$\log_a(b) = \frac{\ln(b)}{\ln(a)} = \frac{\log_z(b)}{\log_z(a)}.$$

- Mention (log-)log-plots at some point in time. (This will be done near the end of the ENGR class.) Two possible uses:

  - Compressing scales
  - Undoing exponentials graphically. (Show by plotting 10^x in a log scale on the y-axis and a linear scale on the x-axis.)

- Find $\lim_{x \to 1^+} \left[\ln\left(x^2 - 1\right) - \ln(x - 1)\right]$

Group Work/Examples.

- Simplify

  - $\log_5 125$
  - $\log_{10}(.1)$
  - $6^{\log_6(x^2+1)}$

- Find the domain.

  1. $f(x) = \ln\left(x^2 + 1\right)$
  2. $f(x) = \ln(3x - 1)$
  3. $f(x) = \ln|x|$
  4. $f(x) = \ln\left(e^x - 5\right)$

- Group exercise (need to show example first), maybe think-pair-share: Write as sums, differences or multiples of single logarithms

  1. $\log_2(32x)$
  2. $\log_{10}\left((x + 1)^2\sqrt{x}\right)$

3. $\ln \frac{xy}{z}$
4. $\log_6\left(x^2 + y^2\right)$

- Combine the following into a single logarithm (show)

  1. $\ln(x) + 2\ln(x+1) - 5\ln\left(x^2+2\right)$
  2. $\dfrac{1}{2}\log_{10}(a) - 3\log_{10}\left(b^2\right) + 2\log_{10}(c)$

- Find an example that shows $\ln(x+y) \neq \ln(x) + \ln(y)$.

- Is it possible or sensible to define a function $\log_1(x)$?

- To show how logarithms can be used to take unknowns out of the exponent:

  Application: Invest $1,000 at 8% annual interest. How long does it take for the investment to double if the compounding occurs

  1. quarterly
  2. continuously (Group ex., could do this first and let them do the hard one.)

Reading Quiz.

Notable Homework Problems.

Section 7.3. Exponential and Logarithmic Equations

Suggested Time. 1 class period

Lecture/Presentation.

- Solve the equation for x:

  1. $x^2e^x + 2xe^x = 0$
  2. $2^{x+3} = 3^{2x-1}$
  3. $\log_2(x) + \log_2(x-1) = 1$, show where the extraneous solutions come from,
  4. $e^x + \ln(x) = 0$ (solve graphically)

- Wish that I could give a formal solution procedure. Any attempt I have seen says nothing but "isolate the exponential/logarithm before undoing it"

- Check some solutions with GC.

- Model for exponential growth or decay $A(t) = A_0e^{kt}$

  growth: $k > 0$ (bacteria growth, interest compounded continuously), decay: $k < 0$ (radioactive decay, energy loss phenomena).

- Exponential growth/decay patterns occur in

  – Radioactive decay,
  – Growth of bacteria (motivate via constant splitting time),
  – Growth of investments (base near 1)

- Iodine 131 decays from $30\mu g$ to $25\mu g$ in 50.5h.

  How long until only $5\mu g$ are left?

  How long until half is left? (talk about half life and doubling time.)

Group Work/Examples.

- Solve for x

  1. $e^{3x+9} = 2$,
  2. $\ln(x+1) + \ln(x-3) = 1$ (needs symbolic quadratic formula, review opportunity, check domain)
  3. $2^{3x-5} = 3^x$
  4. $2^x - 2^{-x} = 1$ (This one will need some explanation when we multiply with 2^x to get the quadratic in 2^x. Graphical solution also a possibility.)
  5. Solve for y. $\ln(y) + 2x = \ln(c)$.

- An investor has invested money at an interest rate of 10% compounded annually. How long will it take for her/his money to double? (Yes, there is enough information.)

  When will the investor have $100,000? (Not enough info. Give starting capital of $10,000 once they notice.)

- .1g of a strand of bacteria is set on a Petri dish at 6:00pm. At 7:00pm there are .3g on the dish. How much will there be at 9:00am the next day?

Reading Quiz.

Notable Homework Problems.

Section 8.1. Basic Trigonometric Identities and Equations

Suggested Time. 2 class days. One on identities and one on equations.

Lecture/Presentation.

- Intro of trig identities and equations can be motivated via graphs of $a\sin(x)+b\cos(x)$

  Other motivations are the need to change expressions in integrals and in derivatives. When working with derivatives we may also need to solve equations.

  Finally, the additive identity for sine is needed to compute the derivative of the sine function.

- Briefly review the main identities (Pythagoras and tangent being sine over cosine)

- Stress the ideas in the box on p. 272, especially that all (including sec, csc, cot) can and usually should be reduced to sines and cosines

- Emphasize graphical checks

- Show Proposition 8.1.8 graphically as in the figures that illustrate the theorem

Group Work/Examples.

- Let students sketch graphs of $a\sin(x)+b\cos(x)$ on a CAS with a being their day of birth and b being their month of birth (or vice versa), ask them to identify the shape.

- Simplify the expression

  1. $(\sin(x)+\cos(x))^2 - 2\tan(x)\cos^2(x)$
  2. $\sin(x)+\cos(x)+\dfrac{\cos^2(x)}{\sin(x)-\cos(x)}$
  3. $\dfrac{\sin(x)+2}{\sin^2(x)-4}$

- Verify the identity.

  1. $(\sin(x)+\cos(x))^2+(\sin(x)-\cos(x))^2 = 2$
  2. $\dfrac{1}{\sin(x)} - \sin(x) = \dfrac{\cos(x)}{\tan(x)}$
  3. $\sin(x)+\cos(x)+\tan(x)\sin(x) = \sin(x)+\dfrac{1}{\cos(x)}$

- Determine if the equality is an identity or not. If so, prove the identity, if not, solve the equation.

  1. $(\tan(x)+1)(\sin(x)-\cos(x)) = \tan(x)\sin(x)-\cos(x)$
  2. $(\tan(x)+\cos(x))\left(\dfrac{1}{\cos(x)}-\cos(x)\right) = \tan^2(x)(\sin(x)+\cos^2(x))$

3. $\sin^2(x) - \cos^2(x) = 1$
4. $2\cos^2(x) - \cos(x) = 1$

- Solve the equation.

1. $\sin(x) - \dfrac{1}{2} = 0$
2. $6\cos^2(x) - 7\cos(x) - 3 = 0$
3. $2\cos^2(x) + 3 = 9\sin(x)$
4. $4e^x \cos(x) - e^x = 0$
5. $\tan(x) = \dfrac{1}{x}$ in the first quadrant
6. $\sin(x) = 1 - x$
7. $3\arcsin(x) - \pi = 0$

Reading Quiz.

Notable Homework Problems.

Section 8.2. Additive Identities

Suggested Time. 1 class period.

Lecture/Presentation.

- Prove at least one of the additive identities geometrically.

  The proof given is geometrically much more intuitive (in the first quadrant) than the standard proof (cf. Project 8.6.1). The price we pay is the ugly extension to all numbers (which one may want to abbreviate or omit).

  If the product rule has already been covered, it may be worth noting that the identity for sine sort of looks like a product rule and the one for the cosine looks like Pythagoras gone bad.

Group Work/Examples.

- Activity on page ACT-18.
- Find the value of $\cos(75°)$.
- Let students prove the additive identity for cosine using cofunction identities.
- Find the value of $\sin(15°) = \sin(45° - 30°) = \sin(45° + (-30°))$ (also reinforces that we do not need the "difference formulas" that are also often taught)
- Prove the identity
  1. $\sin(2x) = 2\sin(x)\cos(x)$
  2. $\cos(2x) = 2\cos^2(x) - 1$
     (The first two prepare for double angle identities.)
- Solve the equation
  1. $\sin\left(x + \frac{\pi}{3}\right) - \cos\left(x - \frac{\pi}{3}\right) = 0$
     (leads to $\sin(x) = \cos(x)$, which can be squared and solved.
  2. $\arcsin(x) - \arccos(x) = \frac{\pi}{6}$

Reading Quiz.

Notable Homework Problems.

- Exercise 10 are the "difference formulas" that are also often taught
- Exercise 12 shows that once we know the additive identities, there is no need to memorize the cofunction identities, since they can be derived.
- In Exercise 14 the additive identity for the tangent is proved.
- Exercise 15 gives an alternative proof of the continuity of the sine function.

Section 8.3. Derived Identities

Suggested Time. 1 class period.

Lecture/Presentation.

- Prove the double-angle identities or refer to group work on Section 8.2.
- Compute $\cos(120°)$ as $\cos(2 \cdot 60°)$
- Prove the half-angle identities.
- Compute $\sin\left(\frac{\pi}{12}\right)$

Group Work/Examples.

- Activity on p. ACT-19.
- Solve the equation
  1. $\cos(2x) = 2\sin^2(x)$
  2. $\cos(2x) + \cos(x) = 2$
  3. $\dfrac{\cos^2(x)}{\sin^2(2x)} = \dfrac{1}{2}$
  4. $2\cos^2\left(\dfrac{x}{2}\right) = 2 - \cos(x)$
- Prove the identity (check graphically)
  - $2\tan(x) - \sin(2x) = 2\tan(x)\sin^2(x)$
  - $2\sin(x) + \sin(2x) = \dfrac{2\sin^3(x)}{1 - \cos(x)}$
  - $2\cos^2\left(\dfrac{x}{2}\right) - \cos(x) = 1$
- Compute $\cos\left(\frac{\pi}{24}\right)$
- Compute $\sin(105°)$ (using that $\cos(210°)$ is a known value)

Reading Quiz.

Notable Homework Problems.

- Exercise 12 proves the double- and half-angle identities for the tangent function.
- Exercise 13 and 14 are the sum-to-product and product-to-sum identities for sine and cosine. These exercises are put into the section on derived identities, since, while they are not half- or double-angle formulas, they are derived and thus not of primary importance. These exercises could already be assigned in Section 8.2.

Section 8.4. Adding Sine and Cosine Functions

Suggested Time. $\frac{1}{2}$ class period.

Lecture/Presentation.

- Explain that while theoretical results often state solutions (for example of differential equations) as sums of sines and cosines, in practice we can only measure a shifted sine or cosine function.

  Therefore, we need to find a way to translate between sums of sines and cosines and shifted sines or cosines.

- The additive identities translate shifted sines or cosines into sums of sines and cosines. Theorem 8.4.1 is the "reverse translation".

- Prove Theorem 8.4.1.

Group Work/Examples.

- Write the function as a shifted sine function.

  1. $f(t) = 3\sin(2t) + 4\cos(2t)$
  2. $f(t) = -6\sin(5t) + 6\cos(5t)$
  3. $f(t) = \sin\left(t + \frac{\pi}{4}\right) + \cos\left(t - \frac{3\pi}{4}\right)$

Reading Quiz.

Notable Homework Problems.

Section 8.5. The Law of Sines and the Law of Cosines

Suggested Time. 1 class period or more, depending on student preparation and class goals.

Lecture/Presentation.

- The laws of sine and cosine are convenient abbreviations that simplify tasks which otherwise would require tricky arguments with auxiliary lines. The law of cosines also plays an important part in deriving the component form of the scalar product of vectors.
- Prove at least one of the law of sines and the law of cosines.

Group Work/Examples.

- Find the remaining sides and angles in the triangle, using the law of sines.
  1. $a = 5$, $\beta = 25°$, $\gamma = 55°$
  2. $a = 5$, $\beta = 25°$, $b = 4$ (ambiguous)
- Find the remaining sides and angles in the triangle, using the law of cosines.
  1. $a = 3$, $b = 7$, $\gamma = 48°$
  2. $c = 4$, $a = 1$, $\beta = 130°$
- Find the remaining sides and angles in the triangle.
  1. $b = 5$, $c = 6$, $\alpha = 7°$
  2. $c = 7$, $\alpha = 65°$, $\beta = 35°$

Reading Quiz.

Notable Homework Problems.

- Exercise 10 connects the laws of sine and cosine to high school geometry.

Section 9.1. How to Define Tangents and Velocities?

Suggested Time. $1 - 1.5$ class periods. Part of this section can be used as motivation at the beginning of Module 6 or Section 6.4.

Lecture/Presentation.

- Lead-in: tangent to a circle intersects the circle only once.

  Show slopes of tangent lines in different places on the circle.

- Show graph where tangent intersects more than once ... how do we define a tangent line?

- Motivation via secants (show animation, can get it off the web)

- Find the tangent of $f(x) = \frac{1}{1+x^2}$ at $\left(1, \frac{1}{2}\right)$ via chart of slopes as second point approaches 1. (Could also use a numerically simpler example and use this one later. There will be enough quadratics throughout this module to warrant using this one, but maybe not at first.)

x	$m_{PQ} = \frac{\frac{1}{1+x^2} - \frac{1}{2}}{x-1}$
.8	-.54878
.9	-.52486
.99	-.50250
.999	-.50025
.9999	-.50003

- Average vs. instantaneous velocity:

  Police officer: "Sir, I've clocked you going 80mph. Is there anything you wish to say?"

  Driver: "Officer, I've been driving for 4 hours now and my odometer shows I've traveled 200 miles. That means I've gone 50mph in the last 4 hours."

- Define average velocity

- Define instantaneous velocity as the value that the average velocities approach as h approaches 0,

- You drop a quarter from the top of a tower and t seconds later it is $s(t) = 192 - 16t^2$ feet from the ground. What is the quarter's velocity 2 seconds after you drop it?

time interval	avg. velocity
$2 \le t \le 2.1$	$\frac{s(2.1) - s(2)}{2.1 - 2} = -65.6 \frac{ft}{sec}$
$2 \le t \le 2.01$	$\frac{s(2.01) - s(2)}{2.01 - 2} = -64.16 \frac{ft}{sec}$
$2 \le t \le 2.001$	$\frac{s(2.001) - s(2)}{2.001 - 2} = -64.016 \frac{ft}{sec}$
$2 \le t \le 2.0001$	$\frac{s(2.0001) - s(2)}{2.0001 - 2} = -64.0016 \frac{ft}{sec}$
$\vdots$	
instantaneous	-64

- Now draw the function and show the slopes of the corresponding secants and tangents
- define tangent line as the line that goes through the point and whose slope is approached by the slopes of the secant lines
- Process is independent of the point and the function
- Algebraic solution for the velocity example: compute the difference quotient $\dfrac{s(x)-s(2)}{x-2}$.
- Revisit the tangent line of $f(x)=\dfrac{1}{1+x^2}$ at $\left(1,\dfrac{1}{2}\right)$,
- Define the average and instantaneous rates of change.
- Introduce the derivative as $f'(x_0)=\lim_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$. Could do some of the group computations with this formula already in this section.

Group Work/Examples.

- Find the instantaneous velocity of a falling parachutist with $h(t)=-16t^2+5,000$ at $t=2s$.
- Find the equation of the tangent line of $f(x)=\sqrt{x}$ at $x=1$,
- Find the equation of the tangent line of $f(x)=x^2+x+1$ at $a=2$,
- Find the slope of the tangent line of $f(x)=2x^2-4x+3$ at an arbitrary x.
- A patient was administered an antibiotic so that 500mg of the antibiotic are in the patient's bloodstream. The amount is tracked over the next twelve hours. (Done by measuring the concentration of the antibiotic in blood samples and then inferring to the amount of antibiotic in the bloodstream.)

$t[hrs]$	$a[mg]$
0	500
1	486
2	472
3	458
4	445
5	432
6	420
7	408
8	397
9	386
10	375
11	364
12	354

  At what (instantaneous) rate is the patient's body disposing of the antibiotic after 5 hours?

- Estimate the slope of a tangent line from a graph.

- If there is time and this section is covered close to the section on exponentials: Find the slope of $f(x) = e^x$ at an arbitrary x. Use technology to estimate the limit that remains at the end or refer to Lemma 6.10.6.

Reading Quiz.

Notable Homework Problems.

- It could be mentioned that many of the exercises are intended to reinforce the mental connection "derivative-slope of tangent-instantaneous velocity" by forcing the reader to constantly (re-)interpret data.

Section 9.2. The Derivative

Suggested Time. 1 class period. Most of this period could be spent on students working with derivatives. It's very much a review of the previous section, only using the word "derivative".

Lecture/Presentation.

- The derivative $f'(a)$ of a function f at a number a is defined to be $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
- Notations $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =: \frac{df}{dx} =: \frac{dy}{dx} =: y' =: Df(x) =: D_x f(x)$, plus in physics time derivatives are denoted with dots on top of the letter ("Newton's dot-age").
- Recall that this is also the way to compute
    1. slopes of tangent lines,
    2. instantaneous velocities,
    3. instantaneous rates of change
- Weave in that $f(x) = |x|$ does not have a tangent at 0,
- Talk about vertical tangent lines.
- Compare the derivative computed via difference quotients with estimates of the slopes of tangent lines (zoom square or measuring). Could use examples from the previous class.
- Estimate some derivatives off graphs, for example order chosen points by sizes of the derivative at the point,

Group Work/Examples.

- Activity on page ACT-20
- Find the derivative of the function at the indicated point
    1. $f(x) = \sqrt{x} + 2x$ at $a = 2$,
    2. $f(x) = \frac{1}{x^2}$ at an arbitrary a,
    3. $f(x) = -2x^3 + x - 5$ at an arbitrary a,
    4. $f(x) = 3x^2 - 4x - 2$ at an arbitrary a (possibly as demo, showing the mistakes that can occur)
    5. $f(x) = 2x^3 - 2x^2 - 1$
    6. $f(x) = x + \frac{1}{x^2}$
    7. $f(x) = x + \sqrt{3x-1}$ (needs them to take a fraction apart)
- Follow up by asking for slopes of tangent lines, declaring some of these functions position functions and asking for velocities?

Reading Quiz.

Notable Homework Problems.

- Exercise 17 provides the point-slope form of the equation of the tangent line. Students often use this one instead of working out the tangent at $y = mx + b$. Pro: It's faster. Con: It's another formula that circumvents (albeit trivial) algebra.

- In Exercise 18 one sided derivatives are used to determine a criterion for differentiability that is similar to Theorem 6.6.4.

Section 9.3. Differentiable Functions

Suggested Time. 1 class period.

Lecture/Presentation.

- Explain (just like for continuity) the idea behind working with a property rather than working with examples. Knowledge about a property takes care of an infinite number of examples all at once.
- How do we make the local notion of differentiability global?
- f is differentiable at a if and only if $f'(a)$ exists. f is differentiable in an interval I if and only if f is differentiable at every number in the interval.
- Recall the way continuity on a set was defined.
- Want to find graphical consequences of differentiability.
- Find where $f(x) = |x|$ is differentiable, note that y' is not defined at $x = 0$ and that it has a jump discontinuity there. (show picture of derivative of $|x|$)

  Can formally talk about the left and right limit of the difference quotient.
- Prove: If f is differentiable at a, then f is continuous at a.
- f has a vertical tangent at a iff $\lim_{x \to a} |f'(x)| = \infty$.
- Show $f(x) = \sqrt{x}$ as an example.
- Summary: Graphical features that show f is not differentiable at a:
  1. discontinuities,
  2. corners (left sided difference quotient has different limit than the right sided difference quotient)
  3. vertical tangents $\left(\lim_{x \to a} f'(x) = \infty\right)$.
- Rolle's Theorem.

Group Work/Examples.

- Find the places on a graph where the function is not differentiable.
- Determine if $f(x) = \begin{cases} x + 2; & \text{for } x \le 2, \\ x^2 - 2x + 4; & \text{for } x > 2, \end{cases}$ is differentiable at $x = 2$.
- Activity on page ACT-21
- For $f(x) = x^2 - 2x + 4$ find all numbers m in $[0, 2]$ that satisfy the conclusion of Rolle's Theorem

Reading Quiz.

Notable Homework Problems.

- Exercises 12 and 13 continue the logic strand. The idea is to increase students' exposure to the theorems by making them analyze the logical structure.

Section 9.4. The Relationship Between the Function f and Its Derivative f'

Suggested Time. 1 class period.

Lecture/Presentation.

- Since we can compute the derivative at any point, let us define the function

  $$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =: \frac{df}{dx} =: \frac{dy}{dx} =: y' =: Df(x) =: D_x f(x)$$

  for all x for which this limit exists.

- Show the mean value theorem and the theorems that characterize growth behavior with the derivative.

  1. If $f'(x) > 0$ on an interval, then f is increasing,
  2. If $f'(x) < 0$ on an interval, then f is decreasing.

- Graphical problem: From the graph of f sketch the graph of f'.

  Look at the slope at key points first, then connect.

- Could lead in with sketching the graph of f' from the graph of f. Otherwise this should be group work after getting f' from f.

- Find the intervals where f is increasing or decreasing from graphs of f, and of f', note that depending on what data we have we look for turn-around points or zeroes,

Group Work/Examples.

- Group work: From the graph of f', sketch the graph of f.

- Give a table of values and let students estimate the values of the derivative (reconnect with secant lines)

- For $f(x) = 3x^2 - 2x + 5$ find an equation for $\frac{df}{dx}$, illustrate how to check with calculator by graphing both functions.

- Graph a function and its derivative with a CAS (without showing which is which), let students determine which is which.

- Sketch the graph of the function with the given growth behavior

  1. $f' > 0$ on $(-\infty, -2)$ and $(1, \infty)$, $f' < 0$ in $(-2, 1)$, $f'(-2) = 0$, $f'(1) = 0$;
  2. $f' > 1$ on $(-\infty, -2)$ and $(1, \infty)$, $f' < -1$ in $(-2, 1)$, $f'(-2)$, $f'(1)$ not defined (need not disclose the last part);
  3. $f' > 0$ on $(-\infty, -4)$ and $(0, 3)$, $f' < 0$ on $(-4, 0)$ and $(3, \infty)$, $f'(-4) = 0$, $f'(0) = 0$, $f'(3) = 0$, $f(-4) = 0$, $f(0) = -2$, $f(3) = 4$;

4. $f' > 0$ on $(-\infty, -4)$ and $(0, 3)$, $f' < 0$ on $(-4, 0)$ and $(3, \infty)$, $f'(-4) = 0$, $f'(0) = 0$, $f'(3) = 0$, $f(-4) = 0$, $f(0) = 2$, $f(3) = 4$; this one is impossible; explain how if this happens when they graph a function from data they have derived, they will need to hunt down the mistake.

- Activity on page ACT-22

Reading Quiz.

Notable Homework Problems.

- Exercises 18 and 20 continue the logic strand. The idea is to increase students' exposure to the theorems by making them analyze the logical structure.

- Exercise 22 shows that Theorem 9.4.6 cannot be turned into an if and only if, but that there is such a biconditional for nondecreasing and nonincreasing differentiable functions.

Section 9.5. Higher Derivatives

Suggested Time. 1 class period.

Lecture/Presentation.

- The second derivative is the derivative of the derivative and it is denoted $f''(x) = \dfrac{d\frac{df}{dx}}{dx} = \dfrac{d^2 f}{dx^2}$
- Compute the second derivative of $f(x) = 3x^2 - 2x + 5$
- Interpretations:

function	position
first derivative	velocity
second derivative	acceleration
third derivative	jerk

- Motivate concavity by the desire to analyze a more subtle trend (say, a falling stock that is about to catch itself; or, a monitored missile that is accelerating or decelerating).
  1. If $f''(x) > 0$ on an interval, then f is concave up,
  2. If $f''(x) < 0$ on an interval, then f is concave down.
- Careful: All four combinations of concavity and growth behavior can and do occur (draw chart with concavity across, growth behavior down and fill in the sample pictures)

Group Work/Examples.

- Activity on page ACT-23
- Sketch the graph of a function with given growth behavior, given concavity and certain points on the graph (several examples, make one impossible)
- Sketch a possible graph of f from its derivative; note that there are infinitely many functions with the same derivative.
- If possible, sketch the graph from the given data.
  1. $f' > 0$ on $(-\infty, -2)$ and $(1, \infty)$, $f' < 0$ in $(-2, 1)$, $f'(-2) = 0$, $f'(1) = 0$;
- If possible, sketch the graph from the given data (some axes might be distorted).
  1. $\lim\limits_{x\to-\infty} f(x) = \infty$, $f(1) = 2$,
     $f(2) = 3$, $f(3) = 4$,
     $f(4) = 3$, $f(5) = 2$,
     $\lim\limits_{x\to\infty} f(x) = \infty$,

		1		2		3		4		5	
f''	+		+		−		−		+		+
f'	−		+		+		−		−		+

- Graph a function and its derivatives with a CAS (without showing which is which), let students determine which is which.

Reading Quiz.

Notable Homework Problems.

- Exercises 8 and 9 show how concavity can help in an analysis in which data about lower order derivatives is similar.
- I have seen exercises like Exercise 10 elsewhere, but identifying the function and its derivatives from the graphs is a good conceptual exercise.
- Exercise 11 continues the logic strand. The idea is to increase students' exposure to the theorems by making them analyze the logical structure.

Section 10.1. The Power Rule and the Natural Exponential Function

Suggested Time. 1 class period.

Lecture/Presentation.

- Proof of the power rule, at least for positive integer powers
- Any of the early problems suggested for group work can also serve as an example for the applications.
- Show how to check the derivative for consistency through growth behavior with a graphing device.
- Show how to check derivatives with a CAS. Note that the simplification may not be what one expects (especially for more complicated functions).
- Foreshadow the need for the product rule by asking what the derivative of $f(x) = xe^x$ is.

Group Work/Examples.

- The following may be useable as group work or lecture
  - Determine what the derivative of a constant multiple cf of a differentiable function f is.
    Hint. Set up the difference quotient and find a pattern $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.
  - Determine what the derivative of the sum $f + g$ of differentiable functions f and g is.
    Hint. Set up the difference quotient and find patterns $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ and $\lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$.
  - Find the derivative of e^x.
    Hint. Set up the difference quotient and split the e^{x+h}.
- Find the derivative of
  1. $f(x) = 3x^{100}$
  2. $f(x) = 2\sqrt{x^5}$
  3. $f(x) = \dfrac{2}{x} + 5x^{\frac{3}{2}} - 3e^x$
  4. $f(x) = \dfrac{6}{\sqrt{x}} - 4e^x + x$
  5. $f(x) = x^3(3x + 1)$
  6. $f(x) = \dfrac{3x^4 - x^2 + 2}{x^3}$
- Find the equation of the tangent line at the given point a.
  1. $f(x) = 4x^2 - 2x + 1, a = 1$
  2. $f(x) = \dfrac{x^4 - 2x^2}{x}, a = 3$

3. $f(x) = \sqrt[3]{x} + 2$, $a = 2$

- Find where the function has a horizontal tangent line.

  1. $f(x) = 4x^2 - 2x + 1$ (recall what the vertex formula gives us and emphasize that the vertex formula is now obsolete)
  2. $f(x) = x^3 - 2x^2 - 5x + 3$ (note that this was not possible with algebraic methods)

- Find where the function is increasing or decreasing.

  1. $f(x) = 2x^3 + x^2 - 7x + 1$ (note that this was not possible with algebraic methods)
  2. $f(x) = e^x - x$

Reading Quiz.

Notable Homework Problems.

- Problem 16 reenforces that the derivative translates a horizontal velocity into a vertical velocity. This can be used to prepare for the introduction of the chain rule.

- Problem 17 tests how well students understand the graphical connection between f and f' and how well they have internalized the derivatives of polynomials.

Section 10.2. Graphing Functions

Suggested Time. 1 class period.

Lecture/Presentation.

- Recall the various features of a function, outline the graphing procedure, graph some functions or make students do it.

Group Work/Examples.

- Sketch the graph of the function

  1. $f(x) = x^3 + 3x^2 + 1$ (zero is not easy to find without CAS),
  2. $f(x) = 2x^3 - 9x^2 - 12x$
  3. $f(x) = \left(x^2 - 1\right)^2$
  4. $f(x) = 3x^{\frac{1}{3}} - x$
  5. $f(x) = \begin{cases} -2x^3 - 6x^2 + 8; & \text{for } x > -1, \\ x^2 + 4x + 4; & \text{for } x \leq -1. \end{cases}$

Reading Quiz.

Notable Homework Problems.

Section 10.3. Absolute Extrema and Optimization

Suggested Time. 1 class period.

Lecture/Presentation.

- Explain how to find the absolute extrema of a function.
- Show some modeling problems. (Emphasize the approach "define, plan, execute, check" throughout.)
- Could show Example 10.5.10. Squaring the distance function allows us to solve the problem without the chain rule.
- Could show the modeling for Example 10.5.11. This shows that we need a rule for differentiating compositions.

Group Work/Examples.

- Find the absolute extrema of the function on the given interval.
  1. $f(x) = x^3 + 3x^2 - 24x + 4$ on the interval $[0, 4]$
  2. $f(x) = x^4 - 8x^2 + 3$ on the interval $[-5, 5]$
  3. $f(x) = 8\sqrt[4]{x} - x^2$ on the interval $[0, 2]$
- Optimization problems.
  1. A manufacturer wants to make a rectangular box with a square base and no top that has volume $2000 in^3$. Find the dimensions of the box with the smallest surface area.
  2. A farmer wants to build three adjacent rectangular pens side-by-side. $500 ft$ of fence are available. Find the dimensions of the arrangement with the largest possible area.
  3. A $10,000 ft^3$ oil tank is to be in the shape of a right circular cylinder with hemispheres attached on each end. Find the dimensions of the tank with the smallest surface area.
  4. Present or let students work out Example 10.3.3. Let students model Examples 10.5.10 (can be solved) or Example 10.5.11 (foreshadows the chain rule).
  5. If it has not been assigned earlier, Project 1.9.3 can be presented as an authentic example in which simple calculus gives data that is close to the industry specs. Differences in the results can be explained with the more sophisticated design. (Design set up to improve can strength. Design produced with finite element method.)

Reading Quiz.

Notable Homework Problems.

Section 10.4. The Product and Quotient Rules.

Suggested Time. 1 class period

Lecture/Presentation.

- Lead-in: What is the derivative of $f(x) = e^x(1-x)$?
- Class could be separated into a theory part (proofs) and an application part (differentiate functions, find tangent lines, determine growth behavior, concavity, graph functions etc.)
- Proof of the product or quotient rule, could also be done as an activity (cf. page ACT-3)
- Memory aids.
  - The product rule is just the numerator of the quotient rule with a "$+$" instead of a "$-$".
  - The quotient rule is "(hi)d(lo)-(lo)d(hi)/(lo)(lo)" (seems childish, but it works; macho faculty could demonstrate with shadow-boxing punches or kicks).
- Any of the early problems suggested for group work can also serve as an example for the applications.
- Check derivatives of presented examples as suggested in Remark **??**.
- Check derivatives symbolically with a CAS. Note that the computer's simplification may not be what one expects (especially for more complicated functions).
- Foreshadow the need for the chain rule by asking what the derivative of $f(x) = e^{x^2}$ is.

Group Work/Examples.

- Activity "Differentiating products" on page ACT-3. If the proof of the power rule was presented, the technique should be familiar to students.
- Find the derivative of the function and simplify as much as possible
  1. $f(x) = xe^x$
  2. $f(x) = \dfrac{x+1}{x-1}$
  3. $f(x) = \dfrac{x^2+2x-1}{3x^3-12}$
  4. $f(x) = \dfrac{xe^x}{x^2+2}$
- Find the second derivative of the function and simplify as much as possible
  1. $f(x) = xe^x$
  2. $f(x) = \dfrac{e^x}{x^2}$

- Find the equation of the tangent line at the given point a.

  1. $f(x) = \left(2x^{\frac{2}{3}} + 1\right) e^x$, $a = 1$
  2. $f(x) = \dfrac{2x^4 - 2x^2}{3x^2 - 1}$, $a = 3$

- Find where the function has a horizontal tangent line.

  1. $f(x) = \dfrac{e^x}{2x}$
  2. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

- Find where the function is increasing or decreasing.

  1. $f(x) = \left(x^2 + 1\right) e^x$
  2. $f(x) = \dfrac{x + 1}{x - 1}$

- Sketch the graph of the function

  1. $f(x) = e^x(1 - x)$

Reading Quiz.

Notable Homework Problems.

Section 10.5. The Chain Rule

Suggested Time. 1 class period

Lecture/Presentation.

- Class could be separated into a theory part (proofs) and an application part (differentiate functions, find tangent lines, determine growth behavior, concavity, graph functions etc.)
- Lead in with something like "How do we take the derivative of $\left(x^2 - 5x\right)^{100}$?"
- Proof of the chain rule, could also be done as an activity similar to how the product rule was derived.
- Explain the chain rule via velocities. If I run/drive up a hill at horizontal unit speed, my vertical gain is $f'(t)$ per unit time. If I do the same at horizontal speed g', my vertical gain is $f'(g(t))g'(t)$ per unit time.

  Could work this out with numbers that are realistic slopes for hills and speeds for runners/cars.
- Any of the early problems suggested for group work can also serve as an example for the applications.
- For higher derivatives of quotients it is reasonable to point out that the degree of the numerator will only increase by 1 for each derivative. The first problem in the "second derivatives" group work problems is helpful here.
- Check derivatives with a CAS and/or with a graphing device. Note that the simplification may not be what one expects (especially for more complicated functions).
- Foreshadow the need for rules for trigonometric and inverse functions by reminding students of these functions.

Group Work/Examples.

- Find the derivative
  1. $f(x) = \left(2x^3 - 4\right)^{100}$
  2. $F(x) = \sqrt{x^2 + 1}$
  3. $F(x) = e^{2x^5 - 4x}$
  4. $F(t) = \sqrt{\dfrac{5t^4 + 1}{2t^2 + 3t + 9}} + 7t$
  5. $f(x) = e^{\left(x^2+1\right)^4}$
  6. $f(x) = x^3 e^{x^2}$
- Find the second derivative (can also be cast as "Find where the function is concave up or down.")
  - $f(x) = \dfrac{x^2 + 1}{2x^2 - 1}$ (Note here that correct cancelation is important.)

  - $f(x) = xe^{2x}$

- Find where the function has a horizontal tangent line.
  - $f(x) = e^{4x^2-3x+1}$
  - $f(x) = \sqrt[3]{x^2 - 1}$

- Find the equation of the tangent line at the given point.
  - $f(x) = xe^{2x-6}$, $a = 3$.

- Sketch the graph of
  1. $f(x) = x^2e^{-x}$
  2. $f(x) = \dfrac{x}{x^2 + 1}$
  3. $f(x) = \left(x^2 - 1\right)^2$
  4. $f(x) = x^{\frac{2}{3}}(x - 2)^2$
  5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

- Use the chain rule on $\dfrac{f}{g} = fg^{-1}$ to prove the quotient rule.

  Motivate the idea of all the back-and-forth proofs to show the interrelationships of these rules. If the quotient rule was not proved yet, then this is a way to fill the gap. It can also be seen as a memory aid in case one has a lapse (say one does not remember which term is negative in the quotient rule).

Reading Quiz.

Notable Homework Problems.

Section 10.6. The Derivatives of the Trigonometric Functions

Suggested Time. 1 class period.

Lecture/Presentation.

- Prove Lemma 10.6.1 or show it graphically. Prove the derivative formulas. The activity on page ACT-24 does this also. Examples etc. as in group work section. (Activity does not have any optimization problems.)

Group Work/Examples.

- The activity on page ACT-24 is pretty much a layout for the class, including sample problems from this guide.
- Prove that $(\sin(x))' = \cos(x)$ with difference quotients.
- Find the derivative of $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$
- Find the derivative of $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
- Find the derivative of
  1. $f(x) = \sin^2(3x + 2)$
  2. $f(x) = x\sin(2x)$
  3. $f(x) = x\cos(x)e^x$
- Find where the function is increasing or decreasing
  1. $f(x) = \cos^2(x)$
  2. $f(x) = e^{-x}\sin(3x)$
- Find the 34^{th} derivative of $\cos(x)$.
- Find the n^{th} derivative of $\sin(x)$.
- Find the limit
  1. $\lim_{x\to 0} \dfrac{\sin(4x)}{x}$
  2. $\lim_{x\to 0} \dfrac{\cos(7x) - 1}{12x}$
- Sketch the graph of the function
  1. $f(x) = e^{-x}\sin(3x)$
  2. $f(x) = \dfrac{\cos(x)}{2 + \sin(x)}$ on $[0, 2\pi]$
- (Similar to problem 13.) A $12in$ wide piece of sheet metal is to be turned into a gutter with trapezoidal cross section by bending up $3in$ wide sections on each side. Denote by α the angle between the bent up parts and the horizontal ground. Find the angle that gives the largest possible cross section.

Reading Quiz.

Notable Homework Problems.

Section 10.7. Implicit Differentiation.

Suggested Time. 1 class period

Lecture/Presentation.

- Lead in with the desire to find the slope or the equation of the tangent line of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at an arbitrary point.
- When it comes down to the brass tacks, implicit differentiation is a simple symbolic task. "Pretend y is a function like $f(x)$ and differentiate both sides", similar to what is done for example in Exercise 14 in Section 10.4.
- Advantage of implicit differentiation: It's often quicker than solving for y and then differentiating (if that's even possible). Disadvantage: We need x and y to compute y'.
- Derivatives of inverse functions in general.
- Derivatives of the inverse trigonometric functions via implicit differentiation of equations like $\sin(y) = x$, etc. or via Theorem 10.7.5 (also possible as group work).
- Derivative of the logarithm function via implicit differentiation of $e^y = x$ or via Theorem 10.7.5 (also possible as group work).
- Prove the general version of the power rule.

Group Work/Examples.

- Find the derivative y' at an arbitrary point
  1. $x^3 + y^3 = 6xy$ (Folium of Descartes)
  2. $-6y = x^2 + y^2 + 1$ (See problem below. This is the contour for $h = \frac{1}{2}$.)
  3. $x\cos(y) + y\cos(x) = 1$
- (Primitive lead-in to related rates.)

  Boyle's Law states that at constant temperature and molarity, pressure and volume of an ideal gas are related by the equation $pV = k$, where k is a constant.

  When is the pressure rising faster? When $\frac{dV}{dt} = -2\frac{cm^3}{min}$ and $V = 100cm^3$ or when $\frac{dV}{dt} = -1\frac{cm^3}{min}$ and $V = 50cm^3$.

  Could allude to a bicycle pump or so to explain Boyle's law: If the valve does not open up, we will keep compressing the gas in the cylinder and the increased pressure slows down our compression. (This is not quite accurate, because the temperature of the gas will not stay constant in the fast compressions with a pump.)
- The height of the surface of a mountain is given by $h(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$ (pick your unit).

  North-south is aligned with the y-axis. Where does the contour $h(x, y) = \dfrac{1}{2}$ run straight east-west?

- Let students find the derivative of arcsin, arccos, arctan or ln via implicit differentiation. We might as well be unapologetic that implicit differentiation allows us to derive formulas we will need later. The nature of this module is symbolic.

  Lead in by asking for y' when $x = \sin(y)$.

- Find the derivative of the function

  1. $f(x) = x\ln(x)$
  2. $f(x) = \ln\left|\dfrac{x+1}{x-1}\right|$
  3. $f(x) = \log\left(x^2+1\right)$
  4. $f(x) = \ln(x\sin(x))$
  5. $f(x) = x^{\frac{1}{x}}$ (take logs on both sides or directly)
  6. $f(x) = \ln\left(x^2+5\right)\sqrt{3x+2}$ (Explain with a graphic as in Figure 10.42.)
  7. $f(x) = \arcsin(2x+1)$
  8. $f(x) = \arctan\left(e^x\right)$
  9. $f(x) = \arccos\left(x^2\right)$
  10. $f(x) = (\arccos(x))^2$
  11. $f(x) = x\arcsin\left(x^3\right)$

Reading Quiz.

Notable Homework Problems.

- Problem 17 shows that the product rule is a consequence of the chain rule as long as the functions do not have zeroes.

Section 11.1. Optimization When Parameters are Involved

Suggested Time. 1 class period.

Lecture/Presentation. Optimization problems are challenging and putting in parameters adds another layer to the challenge. As a preparation for multivariable calculus, applied classes and also later work in single variable calculus (volumes of objects for which the dimensions are not numbers) this section can go a long way though. Typically parameters are not emphasized in an extra section in calculus. However, students are expected to work with equations like $pV = \nu RT$, $F = G\frac{mM}{r^2}$ in science classes that are taught in the freshman and sophomore years. By emphasizing parameters a bit more, students overall competence with more complex models should increase.

- Emphasize the "define-plan-execute-check" approach. Aside from the fact that some numbers have turned into parameters, nothing has changed.
- Motivate why parameters are useful by telling a story about a customer (buying boxes that you make) who keeps changing his mind on what materials to use for top, bottom and sides (which changes costs and thus the optimum shape). "... You compute the optimal shape based on the customer's specifications. That takes a while. Just as you have finished the computation for these costs, your customer calls with a new set of costs. (repeat as necessary) ..." Students realize quickly that there has to be a better way.
- The analysis of extreme configurations can serve as a double check against intuition. For example, a box for which the sides cost (per area) a lot more than the top and the bottom should be flat and wide.

Group Work/Examples.

- (This is Example 11.1.4.) A rectangular box with square base and no top is to be made from two types of material. The material for the bottom costs b cents per square inch and the material for the sides costs s cents per square inch. Find the dimensions of the box that costs the least. Then determine what happens to the shape when $b \gg s$.

  Could also use cylinders with and without top.

  Could also give students several sets of costs to apply like in some of the homework problems.

- (Modification of Example 10.5.11.) A new house outside of town needs to be connected to the electrical (or: sewer, phone, cable TV) network. Laying the connection along the roads would require to go 1mi along a highway and then – after a 90 degree turn – $\frac{1}{4}$mi along the rural road that leads to the house. It costs h per mile to lay the connection along the highway and c per mile to lay the connection cross country or along the rural road. What is the cheapest way to connect the house to the network?

  (Could be done with a lead in on several companies bidding on the same project. Also, I think I have seen wires laid that way to houses on the midwestern prairie.)

  How does the connection look when $h \gg c$? How does it look when $c \gg h$?

Reading Quiz.

Notable Homework Problems. The problems in this section can all be considered challenging because of the involvement of parameters. Therefore, the following can help in instruction.

- Let students know that the problems are challenging. It is not abnormal for a single problem to take longer than usual. Some students have no problem getting the computation done. They just get worried because "this should not take so long". We have to get students acquainted with the fact that mathematics problems can and usually do take a long time to solve.
- Students should understand that the answers will be in terms of the parameters. The answers rarely will be numbers.
- One way to start into using parameters is to mimic an example that is worked out with numbers instead of parameters and just replace corresponding numbers with parameters. Some homework problems have references to similar problems in which the parameters are numbers.

Problem specific remarks.

- Exercises 13 and 14 train exactly the kind of modeling that makes it hard for students to compute volumes of square and round pyramids with calculus. The key is to set up the equation of a straight line through two points that are given symbolically. If this is trained here for the first time, later computations in the section on volumes should go better.

Section 11.2. Graphing functions.

Suggested Time. $1-2$ class periods.

Lecture/Presentation. Graphing function requires good organizational skills. Being organized is neither boring, nor tedious, nor uncreative. It simply helps when working with complex tasks. Graphing functions that depend on parameters by hand is a task that can become arbitrarily complex and symbolically tedious, depending on the function chosen. Hence the presented examples and the assigned homework problems should be chosen carefully. I am still looking for good problems here, simply because this task is so challenging.

- Recall the graphing procedure from Algorithm 10.2.1. It is independent of how complicated the function is and also of whether we are working with a family of functions.
- Note that features of complicated functions can still be found using a CAS.
- Give examples of parameter dependent families of functions in applications: probability density functions (such as the normal densities discussed in the section or the Maxwell densities in Exercise 9), pressure vs. volume curves that depend on additional parameters as in the van der Waals equation (cf. Exercise 8),
- Introduce the idea that a parameter dependent family of functions has features whose position (and existence) depends on the parameter. Demonstrate with any of the notable homework problems and suggested group work.

  Graphical exploration (CAS) should precede analysis here.

Group Work/Examples.

- Sketch the graph of the given family of functions. Each example could also be broken up into separate problems "find where the family is increasing or decreasing" and "find where the family is concave up or down".

  1. So far I have most successfully used Example 11.2.4.
  2. Simple example: $f_a(x) = x^2 + ax$
  3. $f_a(x) = x^3 + ax$ (similar to homework problems, but with slightly uglier values for the extrema)

Reading Quiz.

Notable Homework Problems. Many problems in this section are challenging because of the involvement of parameters. Therefore, the following can help in instruction.

- Let students know that the problems are challenging. It is not abnormal for a single problem to take longer than usual.
- Students should understand that the answers will be in terms of the parameters. In particular, an answer may depend on whether a parameter is within a certain range, for example, whether the parameter is positive or negative.

Problem specific remarks.

- Example 7 is a simple "read the parametric graph" exercise, but it can present astonishing challenges. Thinking in terms of parameters simply takes time to develop, so even tasks that experts consider "easy" need to be trained.

- Exercise 8 gives the van der Waals equation (a correction to the universal gas law) as a source for parameter dependent functions. This is also a good connection to freshman chemistry.

- Exercise 9 gives the distribution of molecule velocities in a gas as an example of a parameter dependent function.

- Several families of functions occur in Exercises 4, 5 and 6. In this fashion the larger problem of graphing the family can be broken up into steps.

Section 11.3. Related Rates.

Suggested Time. 2 class periods. The modeling process is subtle and invites errors.

Lecture/Presentation.

- Related rates are basically an application of implicit differentiation.
- Setting up the equation that models the phenomenon might come a bit easier for students who have gone through Section 11.1 already.
- Aside from setting up the model itself, the most important task in related rates is to keep track of which quantities depend on the independent variable and which do not.
- It is important to realize that one should first do all symbolic computations and then substitute in values. Early substitution leads to mistakes because quantities that actually are variable could be interpreted as constant (cf. Exercise 30).

Group Work/Examples.

- Ship A is 50 yards northwest of ship B. Ship A is going south at 3 yards per second. Ship B is going east at 2 yards per second. Is the distance between the ships increasing or decreasing?
- A flying model rocket currently has mass $\frac{1}{2}kg$ and velocity $20\frac{m}{s}$. If the kinetic energy of the rocket is increasing by $10\frac{J}{s}$ and the velocity is increasing by $5\frac{m}{s^2}$, find the rate of change of the mass.
- A $1m^3$ $(= 1,000l \approx 264.2gal)$ gas tank is being filled. Currently the temperature of the gas is $290K$ $(= 17C \approx 63F)$ and the tank contains 50 moles of the gas at a pressure of $120,495\frac{N}{m^2}$ $(\approx 1.19atm)$. If the fill rate is $10\frac{mol}{min}$ and the pressure change is $20,000\frac{N}{m^2 min}$ $\left(\approx .2\frac{atm}{min}\right)$, is the tank heating up or cooling down?

  $pV = \nu RT$, $R \approx 8.31\frac{J}{K \cdot mol}$

  Can talk about predicting the behavior of a filling process (heating) or the needs (if given parameters are required, cooling may be needed). Can talk about monitoring temperature even if there is no thermometer in the vicinity.

  Same problem is a suggested problem for Section DMV.D.

  Don't overdose on problems for which the formula is given. The geometry is the hard part for related rates. (Problems like that are below.)

- Dr. Absentminded is fishing and finally has a fish on the hook. Dr. Absentminded is reeling in the fish at a rate of $\frac{1}{2}\frac{m}{s}$. The tip of the rigid, vertical fishing pole is $2m$ above the water line and the fish is at the surface of the water. How fast is the fish approaching the fisherman when they are $5m$ apart?

- A funnel is shaped like a right circular cone with top diameter $12in$ and height $16in$. Water is filled in at a rate of $10\frac{in^3}{s}$ and it is exiting at a rate of $7\frac{in^3}{s}$. How fast is the water level in the funnel rising when the water level is at $5in$?
- A camera is filming the lead car of a parade that takes place on a road $100ft$ away from the camera. How fast is the parade moving when the angle between the road and the line of sight of the camera is $\frac{\pi}{6}$ and it is changing at a rate of $0.1\frac{1}{min}$?

Reading Quiz.

Notable Homework Problems.

- Exercise 30a is Exercise 24 of Section 10.7 solved incorrectly. It may be interesting to assign both on the same homework.

Section 11.4. L'Hôpital's Rule.

Suggested Time. 1 class

Lecture/Presentation.

- Motivate l'Hôpital's rule at a point a with limits of numerator and denominator being zero through replacing the function with its tangent line. $\dfrac{f(x)}{g(x)} \approx \dfrac{f'(a)(x-a)}{g'(a)(x-a)}$, so the limit of the quotient should have something to do with the quotient of the derivatives.
- Note that the quotient rule and L'Hôpital's rule must not be confused.
- Explain how to handle the various types of indeterminate forms via examples.

Group Work/Examples.

- The activity on page ACT-27 can be used to pretty much structure the class. Motivate by talking about indeterminate forms. Introduce l'Hôpital's rule once the first column is done.
- Find the limit
  1. $\lim_{x\to 0} \dfrac{\sin(4x)}{9x}$
  2. $\lim_{x\to 0} \dfrac{\cos(x)-1}{x^2}$
  3. $\lim_{x\to \infty} \dfrac{x}{e^x}$
  4. $\lim_{x\to 0} \left(\dfrac{1}{x}\right)^x$
  5. $\lim_{x\to \infty} \left(1+\dfrac{1}{x}\right)^x$
  6. $\lim_{x\to \infty} (1+x)^{\frac{1}{x}}$
  7. $\lim_{x\to \infty} \ln(x) - \ln(x+2)$
  8. $\lim_{x\to \infty} \ln(x) - \ln(5x)$

Reading Quiz.

Notable Homework Problems.

- Exercise 12 connects to work with parameters and it also shows that the limit of an indeterminate form can literally be anything.
- Exercise **??** is an attempt to foreshadow Taylor polynomials.

Section 11.5. Error Estimates and Linear Approximations.

Suggested Time. 1 class period

Lecture/Presentation.

- If the input variable of the function $f(x)$ is only known up to a certain error, say as $x \pm \Delta x$, then the error of the output is $\Delta y \approx \left|f'(x)\right| \Delta x$, because $\dfrac{\Delta y}{\Delta x} \approx \left|f'(x)\right|$ (here the Δs denote the errors, which are always absolute values).

  (The differential is $dy = f'(x)dx$.)

- Define absolute and percentage error.

- Lead into linear approximations with an estimate of $\sqrt{82}$ via linearization around $a = 81$.

- Define the linearization of the differentiable function f at a as $L(x) = f(a) + f'(a)(x-a)$.

- Note that $\sin(x) \approx x \approx \tan(x)$ for $x \approx 0$ is often used in Physics.

- Possible bridge to Taylor polynomials. (Not sure how effective this is.)

  The linear approximation for the cosine function is unsatisfying.

  Show that $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = -\frac{1}{2}$. This means that for x near zero we have

  $$\begin{aligned} \frac{\cos(x)-1}{x^2} &\approx -\frac{1}{2} \qquad \text{or} \\ \cos(x) &\approx 1 - \frac{1}{2}x^2 \end{aligned}$$

  Show that there is a good graphical match. Then show that $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1+\frac{1}{2}x^2}{x^4} = \frac{1}{24}$, which means that for x near zero we have

  $$\begin{aligned} \frac{\cos(x)-1+\frac{1}{2}x^2}{x^4} &\approx \frac{1}{24} \\ \cos(x) &\approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \end{aligned}$$

  Show that there is again a good graphical match.

- Questions brought up by the above. It seems the above approximation gets better as the procedure continues.

  1. How do we formalize the procedure?
  2. If we could run the procedure indefinitely, would we get all of the cosine function?
  3. Since running the procedure indefinitely requires summing infinitely many corrections, how do we sum infinitely many numbers?

Group Work/Examples.

- The period of a given AC current has been measured as $T = 0.02s \pm 0.001s$. Estimate the frequency and the error in the estimate.
- A spherical oil tank has a radius of $r = 3ft \pm .01ft$. Estimate the volume of the oil tank and the error in the estimate based on the uncertainty in your measurement.

  Then compute the percentage error of the radius and of the volume.
- Is it reasonable to use 36 as the "base point" when trying to estimate $\sqrt{82}$?
- Find the linearization of $f(x) = \dfrac{1}{x}$ at $a = 1$.

  Then determine the interval in which the linearization is within 0.1 of the function.
- Find the linearizations of $f(x) = \sin(x)$ and of $f(x) = \tan(x)$ around $a = 0$.

Reading Quiz.

Notable Homework Problems.

Section 12.1. Eliminating Variation By Examining Small Increments

Suggested Time. $1 - 1\frac{1}{2}$ class periods

Lecture/Presentation.

- Underlying idea for this section: Over sufficiently small intervals of the independent variable, the dependent variable can be considered to be roughly constant. This is of course the idea that underlies calculus as well as a large number of modeling processes in science and engineering. The deeper students internalize this idea, the easier multivariable calculus and other applied classes will be. The typical problem that I have observed is that students often have trouble seeing the forest for all the trees.

- For introductions via area.

  1. Take stock of the objects for which we can compute the area with a formula (rectangle, triangle, circle, parallelogram)
  2. Note that aside from the circle, the basis for everything is the rectangle. (Triangle is half a rectangle, parallelogram is a sheared rectangle.)
  3. Derive some more area formulas (regular trapezoid, pentagon, hexagon)
  4. Is there a specific formula (that is worth remembering) for the area under $f(x) = x^2$ from $x = 0$ to $x = 2$? (No.)
  5. Approximate the area under $f(x) = x^2$ from 0 to 2 with 4 rectangles. Use right endpoints for the height.

     Is this a good approximation? Could liken it to a land purchase. If this is the measurement done, would you pay for that much area?
  6. How can the approximation be improved? Note that suggestions about somehow approximating the overshoot lead us back to the same problem we already have: How do we find irregularly shaped areas?

     Lead in to the idea of using more rectangles.
  7. Find the area under $f(x) = x^2$ from 0 to 2 as a limit of Riemann sums. (Use the magic three dots for the first and last time, if at all.)

     Go through it step-by-step: What is a, what is b, what is x_k if we use right endpoints?

     Convert to summation notation and show evaluation on a CAS.

     Keep the summation formula around for later comparison.

- For introductions via velocity.

  1. A rock is dropping off a cliff. Its downward velocity is measured every half second as $0\frac{\text{ft}}{\text{s}}$, $16\frac{\text{ft}}{\text{s}}$, $32\frac{\text{ft}}{\text{s}}$, $48\frac{\text{ft}}{\text{s}}$, $64\frac{\text{ft}}{\text{s}}$, $80\frac{\text{ft}}{\text{s}}$, $96\frac{\text{ft}}{\text{s}}$.

     Estimate how far the rock has fallen in the first three seconds. Explain if your estimate is an over- or an underestimate.
  2. Suppose we know that the downward velocity is $v(t) = 32t$. Estimate how far the rock has fallen in the first 3 seconds.

     Use n time intervals, let students determine the length of each interval, the left or right endpoint of each interval, the distance traveled in the k^{th} interval and the total sum. Use a CAS to find out what happens for large n.

     Keep the summation formula around for later comparison.

3. Draw the graph of $v(t)$ and represent the distance traveled as an area to connect to areas.

- For introductions via volumes.

  1. The frustrum of a right circular cone has height 1m, bottom radius 2m and top radius 1m. Approximate the volume of the frustrum.
  2. Find the shape of the cross sections, determine the volume of a thin slice.
  3. Find the radius of the cross section at height h. Use it to find the volume of a thin slice.
  4. Sum up the small volumes. Evaluate with a CAS for increasing values of n.

     Keep the summation formula around for later comparison.

- For introductions via more complicated work, talk students through Example 12.1.7.

  Keep the summation formula around for later comparison.

- Formally define Riemann sums. Show how they reflect what we have done throughout.

- Give a situation in which the force at a certain distance is given by a graph and let students approximate the amount of work done. Could couch it into a tug-of-war story.

  A tug-of-war team pulls the opposing team 10m towards the middle with a force of $F(x) = 4000 - 100x$ Newtons. Approximate the amount of work that was done.

  Is there a way to compute the work geometrically?

Group Work/Examples.

- Take from the lecture context as appropriate. Students can do problems on distance, area, volume, work or parts of these problems as they are presented. Next section will be mostly group work.

Notable Homework Problems.

- Project 12.3.1 can be assigned after this class. It can be motivated as a "program that does large chunks of the homework" for this section and the next.

- Exercise 1 requires students to decompose everyday shapes into simpler geometric shapes.

  In [S. Clark, E. Seat and F. Weber, The performance of engineering students on the group embedded figures test, Session T3A-1, Proceedings of the 30th Frontiers in Education Conference, October 2000, Kansas City, MO] it has been shown that the ability to decompose shapes is correlated with performance in engineering. I'm not sure how to use this insight in instruction, because ultimately it may be part of how one's brain is wired, but the observation is important. Whenever we do algebra, we do a similar decomposition into patterns.

- Exercise 2 requires no computation, but good visualization skills.
- Students sometimes find it hard to determine the endpoints of the intervals. Exercise 7 isolates this skill.
- Exercises 11, 12, 13, and 14 force the student to read through the corresponding examples. Exercise 5 of Section 4.2 would be good preparation.
- Exercises 21 and 22 determine how growth behavior influences the estimates made with Riemann sums.

Section 12.2. The Definite Integral.

Suggested Time. $1 - 1\frac{1}{2}$ class periods. About half on setting up and computing Riemann sums.

Lecture/Presentation.

- Recall the motivation via the need for a method to find exact areas under curves
  - We cannot hope to find formulas for the infinite number of possible complicated shapes.
  - We need a method that can be used on all shapes.
  - The method should give better approximations as it is refined.
  - Refinement process should lead to easily manageable formulas, so that we can take a limit to get an exact result.
- Note that the idea of cutting a quantity into little intervals on which we have more manageable formulas is not limited to areas.
  - We know that if velocity is constant, then $s = vt$. If velocity varies over time, this formula is not valid. We obtain $s = \int_a^b v(t)dt$
  - We know that if the acting force is constant, then $W = Fs$. If the force varies along the way, we we have $W = \int_a^b F(s)ds$.
  - Note how in both cases above the distance traveled and the work done can be interpreted as areas.
  - We know that the volume of a cylindrical object is $V = Bh$. If however, the cross-section varies with the height, we obtain $V = \int_0^h C(x)dx$ (object was laid on its side).
- Use the connection to distances to motivate the idea to connect definite integrals with indefinite integrals (foreshadows the FTC).
- Taking the limit is the last step that we have not taken yet.
- Define a Riemann sum and the definite integral. Show animation of Riemann sums that are continuously refined.
  - This method can be used on any shape.
  - It gives better approximations as we increase the number of rectangles.
  - As we let the number of rectangles go to infinity, we have formulas that we can take the limit of.
- Note that while taking the limit is an additional step that also requires computation and effort, there is much more sophistication in the modeling we have done so far.
- In a class with strong theoretical focus, the proof why continuous functions are integrable could be outlined.

- After some computation, note that, while we now can compute areas, a simpler process would be nicer. (Similar to how we went from difference quotients to derivative formulas.)

  Note that this will be done in Module 14 with the Fundamental Theorem of Calculus.

- Linearity of the definite integral.

- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

- Integral theorems for even and odd functions. Even: $\int_0^a f(x)\,dx = \int_{-a}^0 f(x)\,dx$; odd: $\int_0^a f(x)\,dx = -\int_{-a}^0 f(x)\,dx$.

- Reconnect with areas by asking for some integrals under functions that can be obtained geometrically.

  Note how we sometimes actually use areas to visualize or even compute integrals that have other meanings in applications (work, distance, etc.).

Group Work/Examples.

- Find the definite integral (using Riemann sums)

  1. $\int_0^1 x\,dx$
  2. $\int_1^3 x^2\,dx$
  3. $\int_0^2 2x^3 - 8x\,dx$ (Check graph to see that areas are counted negatively.)
  4. $\int_0^5 e^x\,dx$ (If the geometric sum is available.)

- Find the volume of the solid obtained by rotating the function $f(x) = 2 - x$ about the x-axis on $[0, 1]$.

  This is the volume of the frustrum of the cone discussed in the previous class. Let students explain why.

- A tug-of-war team beats the opposing team by pulling them the final 5m towards the middle with a force of $F(x) = 3000 + 200x$, for $0 \le x \le 5$. Find the amount of work that was done.

- Given that $\int_1^3 f(x)\,dx = 4$, $\int_3^5 f(x)\,dx = 9$, and $\int_1^5 g(x)\,dx = 7$, find $\int_1^5 2f(x) - 3g(x)\,dx$.

- Given that $\int_{-1}^2 f(x)\,dx = 9$, $\int_{-2}^0 f(x)\,dx = 5$, and $\int_{-1}^0 g(x)\,dx = -2$, find $\int_{-1}^0 5f(x) + 9g(x)\,dx$. Assume that f is odd.

- Find $\int_1^0 x\,dx$ (Note that we found the one with the bounds reversed already. Use this to motivate that $\int_a^b = -\int_b^a$)
- Find (graphically) the integral $\int_{-4}^{10} f(x)\,dx$ for the function

  $$f(x) = \begin{cases} \sqrt{9-(x+1)^2}+2; & \text{for } -4 \le x \le 2, \\ x; & \text{for } 2 \le x \le 6, \\ 6-x; & \text{for } 6 \le x \le 10. \end{cases}$$

- Find the area under the function $f(x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$
- Find the area under the function $f(x) = \begin{cases} \sqrt{4-(x-2)^2}; & \text{for } 0 \le x \le 4, \\ 2x-8; & \text{for } 4 \le x \le 6, \\ 4; & \text{for } 6 \le x \le 10. \end{cases}$
- Find the area under a function that is given graphically.
- Set up (do not solve) a definite integral for the following.
  - The distance traveled by an object that travels along a straight line with velocity $v(t) = 3t^2 - t$.
    (Sketch the area.)
  - The work done by moving an object along a straight path from $x = 0$ to $x = 10$ if the force required to move the object is $\frac{1}{50}x(x-10)+2$.
    (Sketch the area.)
  - The volume of a solid sphere of radius 2.
  - The work that it takes to empty liquid out of a spherical tank of radius 2 through an opening at the top.
- Estimate
  - Distance traveled from a velocity graph.

Reading Quiz.

Notable Homework Problems.

1. Exercise 12 revisits the modeling problems in the exercises for Section 12.1.
2. Exercise 13 is part of the strand of logic problems through the text. The idea is to make students study the statement of a theorem by doing something with it.
3. Exercises 17 and 18 show what can go wrong with (Riemann) integration if the function is sufficiently pathological.

Section 13.1. Basic Indefinite Integrals

Suggested Time. 1 class period

Lecture/Presentation.

- What is the derivative of the position function of an object? What is the derivative of the velocity function?
- One motivation for indefinite integrals is to recover data on the position from data on the velocity or velocity and acceleration.
- Define antiderivatives/indefinite integrals. Note that antiderivatives are only unique up to a constant.
- Note that we will make the connection to definite integrals in Module 14.
- Show graphically the effect of the integration constant (could weave direction fields in here).
- Derive the power rule for integration. Make a table with the elementary antiderivatives.
- Note that constants and sums behave for antiderivatives just like they do for derivatives.
- Define the direction field for a function. Can liken them to the velocity field in a wind tunnel. The antiderivatives are the streamlines that we would obtain by introducing dye into the flow. Possibly foreshadow direction fields for differential equations and with vector fields.

Group Work/Examples.

- An object moves along a straight path with velocity $v(t) = t$. What is its position after t seconds?

  (Do before disclosing the power rule. Discovery of antiderivative leads into antiderivatives and paves the way for the power rule. Lack of sufficient data to complete the problem leads to the idea of integration constants.)
- Find the antiderivative of $\sin(x)$, $\cos(x)$, e^x, $\dfrac{1}{\sqrt{1-x^2}}$, $\dfrac{1}{1+x^2}$ (could let students generate the table of elementary antiderivatives in Theorem 13.1.9).
- Find the antiderivative of
  1. $f(x) = 3x^2 - 2x^{-4} + \dfrac{5}{x^{\frac{7}{3}}}$
  2. $f(x) = 3e^x - \dfrac{1}{2}\cos(x) + \dfrac{6}{x}$
  3. $f(x) = \sqrt[3]{x^2} - 2x^{-1} + 9\sin(x)$
  4. $f(x) = \dfrac{4}{1+x^2} + \dfrac{3x^4 - x^3}{4x} + \dfrac{7}{\sqrt{1-x^2}}$
  5. $f(x) = e^{-2x} + \cos(5x)$ (foreshadows substitution)

- An object travels along a straight path with velocity $v(t) = \frac{4}{1+t^2}$. The initial position of the object is $s(0) = 3$. Find the position of the object at $t = 10$.

  Possible extension: Where is the object coming to rest as $t \to \infty$?

- An object is freely falling off a $192 ft$ high tower. It has no initial velocity. (And earth's gravitational acceleration is $32\frac{ft}{s^2}$.) How high is the object after $1s$?

  Possible extension: When does the object hit the ground?

- Find the graph of the antiderivative of a function through a given point graphically using the direction field.

Reading Quiz.

Notable Homework Problems.

- In Exercise 2 students must match functions with direction fields. (Not that great, but I have not seen this type of exercise before.)
- In Exercise 12 formulas in an integral table a verified via differentiation. This reinforces the idea that we undo differentiation. It also gets students accustomed to the idea of an integral table. Integral tables will remain useful, because a CAS may not always give the answer that is expected (cf. Project 13.6.1). So, why not also use the integral table in instruction? The integrals listed there are certainly among the most "realistic" integrals we will encounter.
- Exercises 14, 15 and 16 investigate the effect of the integration constants in a variety of ways.

Section 13.2. Substitution

Suggested Time. $1 - 1\frac{1}{2}$ class periods.

Lecture/Presentation.

- Prove the theorem on substitution
- Show how the substitution method with solving for dx is an implementation of the theorem that is easy to remember
- Give some hints on how to find what to substitute. Generally we need to hunt for a composition and make sure that something like the derivative of the function is multiplied to the composition.
- Re-emphasize that we can double check results by differentiating.
- Also show how to double check using a CAS.
- Start discussion of parameter dependent integrals.

Group Work/Examples.

- The activity on page ACT-29 can be used to drive the whole class. Students know how to take derivatives and they know how to follow directions. Thus they should be able to do the discovery part. After that, Theorem 13.2.2 can be discussed and (after an example or two) students can solve the integrals on the right side of the activity.
- The velocity of an oscillating mass attached to a spring is $v(t) = \cos(4t)$. If the mass is at $p(0) = 5$ at time $t = 0$, where is the mass at $t = 3$s?
- For trigonometric tricks and substitution. Find the integral (check all with CAS)
  1. $\int \cos^3(x)dx$
  2. $\int \sqrt{1-x^2}dx$ or $\int \sqrt{r^2-x^2}dx$
  3. $\int \frac{x^2}{\sqrt{9-x^2}}dx$
  4. $\int \cos^2(x)\sin^2(x)dx$ (show how to use the reduction formula in the integral table)
  5. $\int \sqrt{4+x^2}dx$ (substitute $u = 2\tan(x)$ and the show how to use reduction formula, discuss how sensible it would be to use the table directly)
- Solve the integral $\int x\sqrt{a+bx^2}\,dx$
- Solve some of the integrals needed for partial fractions

Reading Quiz.

Notable Homework Problems.

- Exercise 10 lays some groundwork for integration via partial fraction decompositions.
- In Exercise 11 formulas in the integral table are verified by performing the appropriate substitutions. This accustoms the students to performing substitutions when some parameters are involved. This is not only useful to verify the formulas in an integral table. Simple substitutions are also frequently needed to apply the formulas in an integral table.

Section 13.3. Integration by Parts

Suggested Time. $1 - 1\frac{1}{2}$ class periods.

Lecture/Presentation.

- Derive the formula for integration by parts by reversing the product rule.
- "Li Ate" (Logarithmic, Inverse, Algebraic, Trigonometric, Exponential) The first type of function that occurs in this acronym and in the integrand is the function that should be differentiated. It's a simple acronym and many students ignore it. For some, however, it is very helpful.
- Re-emphasize that we can double check results by differentiating.
- Also show how to double check using a CAS.

Group Work/Examples.

- Activity on page ACT-30.
- The vertical velocity of a wheel that went through a pothole is $v(t) = -te^{-4t}$. If the wheel is at $y = 0$ at time $t = 0$, find the position of the wheel at time $t = 1s$.
- Prove formula 35 in the integral table using integration by parts. (It's also a homework problem, but this is potentially the hardest one. Hence it may help to show what to do.)

Reading Quiz.

Notable Homework Problems.

- Exercise 12a is the last integral we need to be able to integrate via partial fraction decomposition.
- Exercise 13 continues the strand of verfying formulas in the integral table. This accustoms the students to working with parameters in integrals.

Section 13.4. Partial Fraction Decomposition

Suggested Time. 1 class period.

Lecture/Presentation.

- Consider the integrals $\int \frac{1}{x^2-1}\,dx$ and $\int \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}\,dx$. Could introduce partial fraction decomposition by asking which is easier to solve. Show that both are equal (maybe with CAS).
- Motivate the desire to integrate rational function via the fact that we already can integrate polynomials. Also could allude to Laplace transforms (and Z-transforms) in which these decompositions play an indispensable role.
- (Re)Introduce the rational functions that we can integrate. (Prior exercises have shown how to integrate these functions.) Note that we will reduce everything to pieces like this.
- Demonstrate partial fraction decomposition with $\int \frac{1}{x^2-1}dx$
- State the full theorem about partial fraction decompositions.
- Further examples
  1. $\int \frac{x^3}{x^2+5x+6}\,dx$ (division and Heaviside's method; could also delay presentation of Heaviside's method until the discussion of the solution of the activity on page ACT-31)
  2. $\int \frac{x-3}{(x-1)\left(x^2+3\right)}\,dx$
  3. $\int \frac{x^2}{(x+2)^2(x-2)}\,dx$
- Partial fraction decompositions will also be of vital importance when solving differential equations with Laplace transforms.
- Setting up a solution of a certain form but with unknown parameters is a common trick. We will also see it frequently when solving differential equations.
- Show how to solve the integrals and the partial fraction decompositions on a CAS.

Group Work/Examples.

- Activity on page ACT-31.

Reading Quiz.

Notable Homework Problems.

- Exercise 4 checks if students can set up a partial fraction decomposition without burdening them with the remaining tedious computations. Many times students' computations here and in Laplace transforms fail because the initial setup is wrong. The problem is with the obvious suspects: Repeated factors and irreducible quadratic factors are not treated correctly.

Section 13.5. Deciding Which Integration Method to Use

Suggested Time. 1 class period. This is also a good "review section" before a test that includes integration. Class could progress to the Fundamental Theorem of Calculus, etc. and then pick up this section shortly before a test.

Lecture/Presentation.

- Give some examples as to how to decide which integration method to choose.
- Give some examples that show how to use an integral table.
- A CAS should have been used to verify computations all along. Even if not, not much of a comment should be needed on using a CAS. You make friends with the way the CAS requires the input and then do the computation. Project 13.6.1 shows some potential pitfalls.

Group Work/Examples.

- Activity on page ACT-32

Reading Quiz.

Notable Homework Problems.

- Project 13.6.1 can be assigned after this class period. Exercise 8 connects to this project by providing logarithmic expressions for inverse trigonometric functions. These expressions are sometimes produced by a CAS instead of the inverse trigonometric function itself.
- Exercise 6 concludes the strand of proving formulas in the integral table.
- If definite integrals are already known, Exercise 7 finds the area of an ellipse via using an integral table.
- If definite integrals are already known, Exercise 10 foreshadows improper integrals.

Section 14.1. The Fundamental Theorem of Calculus – Antiderivative Form

Suggested Time. $\frac{1}{2} - 1$ class period.

Lecture/Presentation.

- Motivate the Fundamental Theorem of Calculus using distance computation via definite integrals $\int_a^b v(t)\, dt$ and indefinite integrals $p(b) - p(a)$. Since the distance does not depend on how it is computed, the two notions of integration must yield the same result. This is the Fundamental Theorem of Calculus.
- In a strongly theoretical class, prove that $\int_a^b f(x)\, dx = F(b) - F(a)$ for continuous f. (Fundamental Theorem of Calculus, Antiderivative Form)
- Show the mechanism of substitution with transforming boundaries in definite integrals. (Though we have back substitution, this comes back to haunt us in multiple integrals.)
- Note that this and the following sections are a good opportunity to use integration methods in context and without any hint which method will work.

Group Work/Examples.

- Find the integral and compare with the results from the class on Riemann sums.
  1. $\int_0^1 x\, dx$
  2. $\int_1^3 x^2\, dx$
  3. $\int_0^3 2x^3 - 8x\, dx$ (Check graph to see that areas are counted negatively.)
  4. $\int_0^5 e^x\, dx$
- Find the integral
  1. $\int_{-2}^3 8 + 4x^2 + x^3\, dx$
  2. $\int_0^2 \sqrt{x}\, dx$
  3. $\int_1^4 3\sqrt{x^3} + \frac{1}{1+x^2} - e^x\, dx$
  4. $\int_0^1 e^{2x}\, dx$
- Find the (oriented) area under the function f in the given interval $[a, b]$.
  1. $f(x) = x\cos(x)$, interval $\left[0, \frac{\pi}{2}\right]$

2. $f(x) = \dfrac{x}{x^2+1}$, interval $\left[1, \sqrt{e-1}\right]$

- Find the volume obtained by rotating the function $f(x) = e^{-x}$ about the x-axis on the interval $[0, 5]$.

Reading Quiz.

Notable Homework Problems.

- Exercises 8, 9 and 10 pick up where we left off in Module 12 and require the students to reactivate the modeling ideas.
- Exercises 14 and 15 are part of the strand of logic problems through the text. The idea is to make students study the statement of a theorem by doing something with it.

Section 14.2. Computing Areas Under and Between Curves.

Suggested Time. 1 class period.

Lecture/Presentation.

- Groundwork for this section was already laid in Module 12.
- Introduce/recall the idea of oriented area with $\int_{-1}^{1} x^3 dx = 0$. This could be done by asking for the area between $x = -1$, $x = 1$, the x-axis and the function $f(x) = x^3$ and first deliberately answering it incorrectly by just setting up the definite integral.
- **Don't present the formulas. Explain and solve problems verbally ("upper function minus lower function...").** Formulas are in the book, but the visualization is much more important here.

  This will come back to us in multivariable calculus. Sooner or later the formulas break down, while the visualization skill remains.
- Explain Theorem 14.2.1 and how it induces the procedure on page 481.

  (Emphasize the need to graph the function.)
- Generalities about the topic.
  - Measuring areas was one of the original motivations for calculus.
  - Forward connection: Double integrals over general regions. The $f(x)-g(x)$ become the bounds of the integrals.
  - How does a good background here affect the performance on this later subject?

    There is correlation (observed in earlier students)
- Area between xe^{x^2}, $x = 0$, $x = 1$ and the x-axis (sketch, show approximation with rectangles)
- Could find formula for the area of a circle, but would need CAS or table for the integral itself.
- Explain Theorem 14.2.8 and how it induces the procedure on page 484.

  (Emphasize the need to graph the functions.)
- Find the area between $f(x) = x^2$, $g(x) = -x + 6$ (sketch, show how we fill it up with rectangles, show that the rectangles are differences of Riemann rectangles, find intersection points for boundaries, then solve.)
- Find the area bounded by $f(x) = \frac{x}{2}$, $g(x) = \frac{1}{\pi}\arcsin(x)$.

  Show that changing to integration along the y-axis helps here.
- Note that this switching of the integration variables will be very helpful in multiple integrals also.

Group Work/Examples.

- Find the area between $f(x) = \sin(x)$, $x = 0$, $x = \dfrac{3\pi}{2}$ and the x-axis.

- Find the area between the function, the two vertical lines (if given), and the x-axis.

  1. $f(x) = x^2 - 1$, $x = -2$, $x = 3$
  2. $f(x) = (x - 2)(x + 1)(x + 4)$
  3. $f(x) = x\sin(x)$, $x = 0$, $x = 2\pi$
  4. $f(x) = 1 - 2\sin^2(x)$
  5. $f(x) = \sin(x)\cos(x)$, $x = 0$, $x = 2\pi$
  6. $f(x) = \dfrac{1}{1 + x^2} - \dfrac{1}{2}$
  7. $f(x) = \cos^2(x)$, $x = 0$, $x = 2\pi$

- Find the area between $f(x) = -x^2$, $g(x) = -x + 6$, $x = 0$ and $x = 4$. Solve integrals by hand, check with CAS.

- Find the area between the two functions and (if given) the two vertical lines.

  1. $f(x) = 2x^2 + 1$, $g(x) = 3 - 3x$
  2. $f(x) = x^3 + 4x^2$, $g(x) = 7x - 10$
  3. $f(x) = \ln(x)$, $g(x) = \dfrac{1 - x}{10}$ (needs CAS for second point of intersection)
  4. $f(x) = \sin(\pi x)$, $g(x) = \dfrac{1}{4}\sin(2\pi x)$, $x = 0$, $x = 1$

- As an area that depends on a parameter, could compute the area of an ellipse (if this is not assigned as homework)

- Compute the area by integrating along the y-axis.

  1. $f(x) = \left(e^2 + 1\right)\ln(x - 1)$, $g(x) = 2x$

Reading Quiz.

Notable Homework Problems.

- In Exercise 10 the area of an ellipse is computed using an integral table for the integrals.

- The result of Exercise 11 is needed in the derivation before Definition 16.2.9.

- Exercises 12 and 13 are part of the strand of logic problems through the text. The idea is to make students study the statement of a theorem by doing something with it.

Section 14.3. The Fundamental Theorem of Calculus – Derivative Form.

Suggested Time. $\frac{1}{2} - 1$ class period.

Lecture/Presentation.

- Note important examples of functions for which we need an antiderivative but cannot find one symbolically. Could show for some of them why integration methods fail. (The examples below focus on the "meat" of the functions. Constants and signs are sometimes omitted.) Could also put them through a CAS to show that the computer does not help either.

  1. $f(x) = e^{x^2}$ (core of the normal distribution)
  2. $f(x) = x^{\frac{1}{2}} e^{-\frac{x}{2}}$ (core of χ^2-distribution with 3 degrees of freedom)
  3. $\dfrac{1}{\left(1+\frac{x^2}{3}\right)^{\frac{3}{2}}}$ (core of the Student's t-distribution with 3 degrees of freedom)
  4. $f(x) = \sin\left(\dfrac{\pi x^2}{2}\right)$ (leads to the Fresnel function)

- Explain that what happens here is mainly a change in point-of-view.
- Derive the derivative form of the Fundamental Theorem of Calculus form the Antiderivative form.
- Show how to get graphs of $F(x) = \int_a^x f(t)dt$ with a CAS.

Group Work/Examples.

- Find the derivative of the function

  1. $F(x) = \int_{-1}^{x} \cos(t^2)dt$
  2. $F(x) = \int_{x}^{0} e^{t^3}dt$
  3. $F(x) = \int_{1}^{2x^2} \dfrac{\ln(t)}{t}dt$
  4. $F(x) = \int_{2x}^{e^x} \dfrac{\sin(t)}{t}dt$

- Give graph of $f(t)$ and ask for the value of $F(x) = \int_0^x f(t)dt$ for various values of x.
- Solve the equation $\int_0^x e^{t^2}dt = 1$ (CAS)

Reading Quiz.

Notable Homework Problems.

- Note that the Derivative Form of the FTC is a last resort. Exercises 2 and 3 are designed to make students determine if it is really needed.

Section 15.1. Volumes.

Suggested Time. $1 - 1\frac{1}{2}$ class periods.

Lecture/Presentation.

- Explain the volume formula verbally. "The volume is the integral over the area of the slices times their (differential) height." Reiterate that the visualization is the most important part here. **Do not present any of the formulas for special cases such as volumes of rotation. Memorization of formulas is the antithesis of visualization.**
- Show that the volume is the integral of the areas of the cross sections along a given axis.
- The setup for a volume computation is a 3-step process.
    1. Decide on an axis through the object. Make this axis your y-axis.
    2. Find the horizontal cross sections, determine their areas depending on the position on the axis (sometimes another cross section along the y-axis helps).
    3. Determine the bounds of the object along the y-axis and solve the integral.

  We could also go along the x-axis if we like that visualization better.
- Derive the formula for the volume of a ball of radius r.
- Prove the formula for the volume of a cylinder with given base.
- Demonstration? Slice an apple to show how we use cross sections?
- For solids of rotation can allude to turned components made on a lathe.
- Also note that we are working more with parameters in this section than previously. This is because we want to find general geometric formulas. (It's also good preparation for physics, multivariable calculus, etc.)

Group Work/Examples.

- Hand out some rotationally symmetric objects, ask students to draw the function that must be rotated.
- Volume of the solid obtained by rotating $y = x^2$ between 0 and 1 about the x-axis.
- Find the volume of the solid obtained by rotating the area bounded by $f(x) = xe^x$, $x = 0$, $x = 1$, and the x-axis about the x-axis.
- Find the volume of the solid obtained by rotating the area bounded by $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ about the x-axis.
- Find the formula for the volume of the cap of a ball of height h,
- Prove the formula for the volume of a square pyramid with base a^2 and height h. (Side length of a cross section is $y = \frac{a}{h}x$ if it is drawn with the tip at the origin and symmetric with respect to the x-axis.)

- Find the volume of the solid that is between $x = 0$ and $x = 4$ and whose cross sections are squares of side length $3x + 2$ (could be used as a lead-in for volume of the square pyramid, more direct with side length x).

- Find the volume of the solid whose cross sections are equilateral triangles with base length $f(x) = \sqrt{1 - x^2}$.

- Activity on page ACT-33. Find the volume of a tetrahedron of side length a.

The hard part is finding the height. Vertical axis goes through the center of the bottom triangle. Put one side on the x-axis, with one corner at the origin. Center of bottom triangle must be on the line $y = \frac{1}{\sqrt{3}}x$, because the line that goes through the origin and through the center has a $30°$ angle of inclination. The x-coordinate of the center is $\frac{a}{2}$, so the center is at $\left(\frac{a}{2}, \frac{a}{2\sqrt{3}}\right)$. Side view. The height is the third side of a right triangle with hypothenuse $\frac{\sqrt{3}}{2}a$ (height of one of the triangles) and for which the other side is $\frac{a}{2\sqrt{3}}$ (the distance of the center point of the base triangle from any of the sides of the base triangle). Thus $h = \sqrt{\frac{3}{4}a^2 - \frac{1}{12}a^2} = a\sqrt{\frac{2}{3}}$.

$$V = \int_0^{a\sqrt{\frac{2}{3}}} \frac{\sqrt{3}}{4}\left(a - \sqrt{\frac{3}{2}}y\right)^2 dy = -\sqrt{\frac{2}{3}}\frac{1}{3}\frac{\sqrt{3}}{4}\left(a - \sqrt{\frac{3}{2}}y\right)^3\Bigg|_0^{a\sqrt{\frac{2}{3}}} = 0 - \left(-\sqrt{\frac{2}{3}}\frac{1}{3}\frac{\sqrt{3}}{4}a^3\right) = \frac{\sqrt{2}}{12}a^3$$

Figure MAN.1 is a template to cut out the sides of an actual tetrahedron.

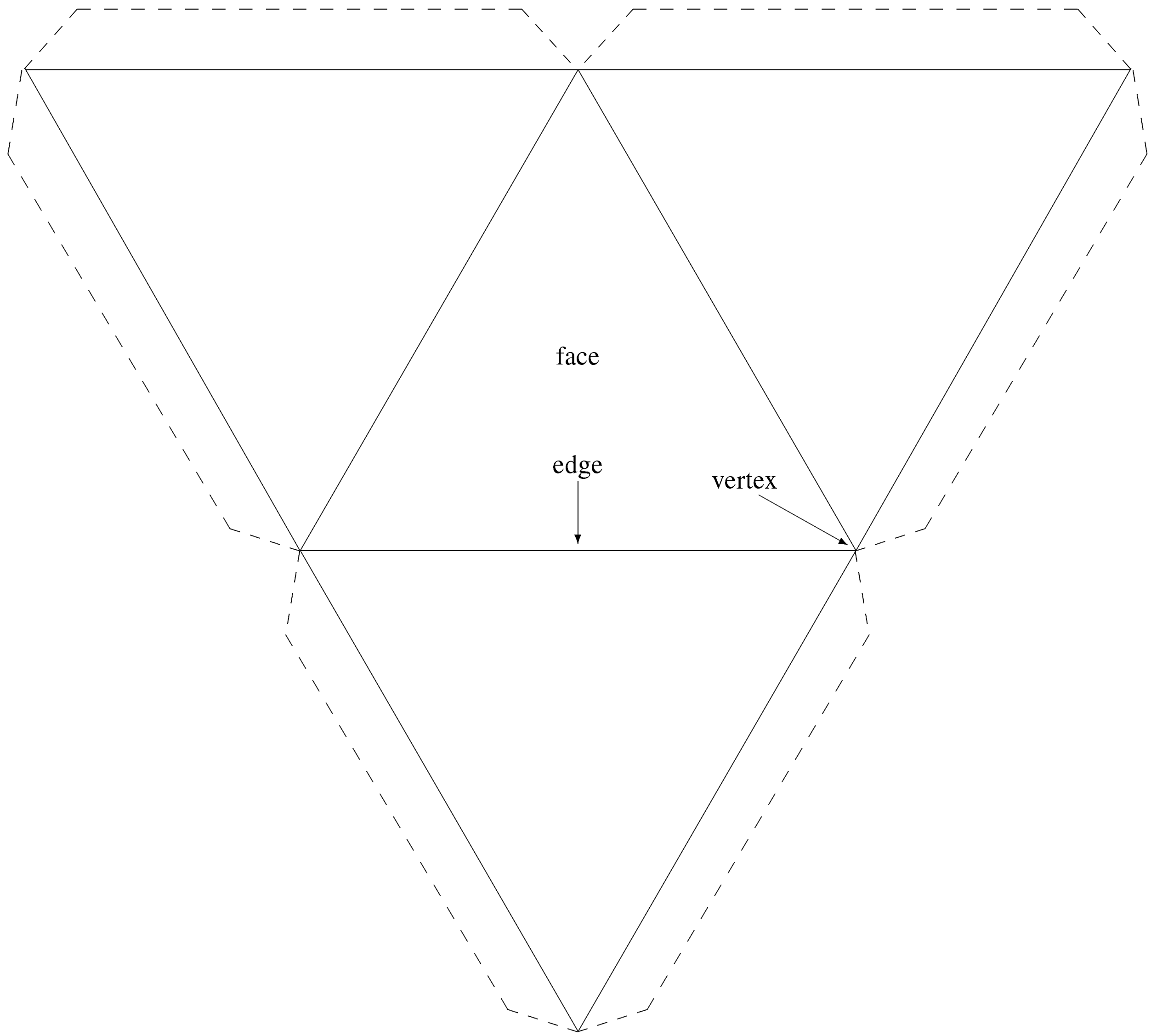

Figure MAN.1: A template for building a tetrahedron.

Reading Quiz.

- To obtain the volume of a solid object we
  - ☐ Integrate the areas of the cross sections.
  - ☐ Multiply base times height.
  - ☐ Integrate the areas of circular cross sections.
  - ☐ Integrate the areas of rectangular cross sections.
- The cross sections in the integral that is set up to obtain a volume must always be
  - ☐ Vertical.
  - ☐ Horizontal.
  - ☐ Perpendicular to the axis along which we integrate.

- Which of the following objects is NOT obtained by rotating a shape around an axis?
  - ☐ Sphere
  - ☐ Torus (donut shape)
  - ☐ Square pyramid
  - ☐ Circular cylinder

- A solid has square cross sections of sidelength $\sqrt{1-x^2}$. What is its volume?
  - ☐ $\frac{2}{3}$
  - ☐ $\frac{2}{3}\pi$
  - ☐ $\frac{4}{3}$
  - ☐ $\frac{4}{3}\pi$

Notable Homework Problems.

- In Exercise 1 the functions that need to be rotated to get a certain object must be drawn.
- Exercise 5 revisits Exercise 2 of Section 12.1. This time the integrals can be solved and the exact number of days can be computed.
- In Exercise 16 the formula for the volume of an ellipsoid is derived using the formula for the area of an ellipse.
- The activity on page ACT-33 could also be assigned as a project.

Section 15.2. Work.

Suggested Time. $1 - 1\frac{1}{2}$ class periods.

Lecture/Presentation.

- Present the definition of work as force times distance.
- Give an example with a variable force, say driving a car on a road through a deep puddle. The engine has to work harder going through the puddle. (Yet the definition still is force times distance.)
- Note that for short distances the force will be constant which leads to $W = \int F\, ds$.
- Compute the amount of work needed to move a 1kg payload to geosynchronous orbit (radius $\approx 42,164$km, which is $\approx 35,787$km above the surface of the earth).

  Compare that with the amount computed via mgh.

  (This is also Exercise 9.)
- Compute the total work needed to pump out a $3m \times 1m \times 1m$ fish tank through the top.

  Or: Compute the total amount of work needed to empty a 10m $\times$ 25m $\times$ 2m swimming pool.

  (Reasonable student suggestion: why not drain the water out the bottom? No plug on the bottom, drain clogged, etc. Could also allude to in-ground oil tanks and use them as an example with appropriate dimensions.)
- Talk about the connection to volume computation. Again we visualize and slice. However, we must usually slice horizontally.

Group Work/Examples.

- How much work is required (at least) to move a 1kg mass 600,000km (approximately two light seconds) away from the surface of the earth?

  Compare this with how much work is required to move the mass 1,200,000km (approximately four light seconds) away from the surface of the earth.
- A drill assembly with linear density $1\frac{kg}{m}$ is pulled out of a $200m$ hole drilled with it. How much energy is required? ("How much work needs to be done?")
- An cylindrical oil tank is $5m$ long and it has a diameter of 2m. The outlet of the tank is at the top. How much work has to be done to empty the tank through the top outlet if it is half full? (Use $920\frac{kg}{m^3}$ as density for oil.)

  Extension. How much work has to be done to empty it if it is 3/4 full? (Need to find the fill height for 3/4 full, which may require us to find the volume of a cut off cylinder lying on its side.)
- Compute the total energy released when a conical water tank of height $1m$ and top diameter $50cm$ is drained through the bottom of the tank.

- Activity. Print the statements below on cards and ask students to put the cards in order. The idea is to get students to formalize their thought process by formally sorting out tasks that they may otherwise not verbalize. Rather than giving the algorithm, they construct it from the pieces, just like they are supposed to construct the solution to an application problem from the pieces of information they know.

  This activity could be done at the start without any further hints. This forces students to reactivate knowledge from Module 12.

  - Make a sketch of the situation.
  - Introduce coordinates as necessary, label quantities.
  - Determine what object(s) is/are lifted.
  - Determine the distance over which the lifted object moves.
  - Determine the volume of the lifted object.
  - Determine the mass of the lifted object.
  - Determine the force that must act on the lifted object.
  - Set up a Riemann sum.
  - Translate the Riemann sum into a definite integral.
  - Solve the definite integral.
  - Perform a unit check.

Reading Quiz.

Notable Homework Problems.

- In Exercises 5 and 6 students show that the work needed to pump out parts of a tank does not solely depend on the volume pumped out.
- In Exercise 8 the energy needed to move a payload to the international space station is compared with the potential energy computed using $W = mgh$.

Section 15.3. Improper Integrals I: Infinite Intervals

Suggested Time. $1 - 1.5$ class periods. This section can be used to further practice integration techniques and also to (re)introduce error estimation via the comparison test.

Lecture/Presentation.

- Motivation: How much energy is needed to escape the gravitational field of the earth? $\displaystyle\int_{R_{\text{earth}}}^{\infty} \frac{G m_{\text{earth}} m_{\text{object}}}{r^2}\, dr$
  Another motivation is statistics. If statistics is covered soon hereafter, then this connection should be highlighted very strongly.
- Give the definition of improper integrals over infinite intervals,
- Work some examples.
  $\displaystyle\int_1^{\infty} e^{-x}\, dx =,$
  show that $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx = \infty.$
  Could culminate in $\displaystyle\int_0^{\infty} x e^{-x}\, dx$ (needs l'Hôpital for limit),
- Show an example of an improper integral that we cannot compute with the fundamental theorem of calculus.
- Introduce the comparison test.
- Show that the integral in the example converges, raise the question of what the value would be.
- Give an example of an improper integral that diverges.

Group Work/Examples.

- The activity on page ACT-34 can be used to lead students through the class.
- Compute the improper integral.
  1. $\displaystyle\int_0^{\infty} e^{-5x} dx =$
  2. $\displaystyle\int_1^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx =$
  3. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2} dx$
  4. $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} dx =$
- Find the values of p for which
  $\displaystyle\int_1^{\infty} \frac{1}{x^p} dx < \infty$
- What is the value of $\displaystyle\int_1^{\infty} e^{-x^2} dx$? Does it even exist?

- Determine if the given improper integral converges. If it does, give a cutoff point so that the integral over the infinite interval beyond this point is less than 10^{-5}.

    1. $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx$
    2. $\displaystyle\int_0^\infty e^{-\frac{x^2}{2}}\,dx$ (good connection if statistics is to be covered soon)

Reading Quiz.

- The integral of a function over an infinite interval.
    - ☐ Is always infinite.
    - ☐ Is always finite.
    - ☐ Can be finite or infinite.
- If a function f approaches infinity at 5, then the area under f from 5 to 6
    - ☐ Is sometimes finite, sometimes infinite.
    - ☐ Is finite.
    - ☐ Is infinite.
    - ☐ Is finite, but cannot be computed.
- Whether an improper integral from 1 to infinity is finite or not depends on
    - ☐ If the derivative of the function has a limit at infinity.
    - ☐ If the function has a limit at infinity.
    - ☐ These improper integrals are always infinite.
    - ☐ If the antiderivative has a limit at infinity.
- The area under the function f(x)=1/x is
    - ☐ Finite from 0 to 1
    - ☐ Infinite from 0 to 1
    - ☐ Finite from 1 to infinity
    - ☐ Infinite from 1 to 100,000

Notable Homework Problems.

- Exercises 12, 13 and 14 are part of the strand of logic problems through the text. The idea is to make students study the statement of a theorem by doing something with it.
- Exercise 8 shows how our basic physics and calculus can be used to compute the event horizon of a black hole.

Section 15.4. Improper Integrals II: Singularities

Suggested Time. $\frac{1}{2}$ class period.

Lecture/Presentation.

- Define improper integrals for functions with singularities. (They arise less frequently than improper integrals over infinite intervals.)

Group Work/Examples.

- Compute the improper integral.

  1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx,$
  2. $\displaystyle\int_0^1 \frac{\ln(x)}{x^{\frac{1}{2}}}dx.$
  3. $\displaystyle\int_0^1 \ln(x)dx =$
  4. $\displaystyle\int_1^3 \frac{1}{\sqrt[3]{x-2}}dx =$
  5. $\displaystyle\int_0^1 \frac{\cos(x)}{x}dx =$

- State and prove the p-integral test for integrals from 0 to a.

- Determine if the given improper integral converges. If it does, give a cutoff point so that the integral over the infinite interval beyond this point is less than 10^{-5}.

  1. $\displaystyle\int_0^1 \frac{1}{(e^x-1)^2}\,dx$

Reading Quiz.

Notable Homework Problems.

Section 15.5. Centers of Mass of Linear Densities

Suggested Time. $\dfrac{1}{2} - 1$ class period.

Lecture/Presentation.

- Present the mass and center of mass formula for linear densities.
- Explain how linear densities arise in wires (primitive model) and in probability theory.
- Explain how the formulas for linear densities can be used to compute the mass and center of mass of objects with a symmetry axis.

Group Work/Examples.

- Compute the mass and the center of mass of the given object.
  - The system made up of a mass $m_1 = 3$kg at $x_1 = 3$ and a mass $m_2 = 5$kg at $x_2 = 7$.
  - The straight wire on $[0, 1]$ with density $\rho(x) = x^2$.
  - A hemisphere (cf. Example 15.5.4) or one of the objects in Exercise 3.

Reading Quiz.

Notable Homework Problems.

Section 16.1.1. Taylor Polynomials

Suggested Time. 1 class, maybe a bit less. This section is broken into polynomials and error estimates because I personally like added emphasis on the error estimates.

Lecture/Presentation.

- Taylor polynomials are a higher-order refinement of the approximation of functions with tangent lines. Tangent line goes through the point and has the same first derivative.

  Goal: Find a polynomial that has the same first k derivatives as the function.

- Prove Theorem 16.1.1.

- Compute the n^{th} Taylor polynomial for $f(x) = \sin(x)$.

- Show how the Taylor polynomials of $\sin(x)$ approximate $\sin(x)$.

  This can be done with an animation in which the degree of the polynomial increases over time.

- Develop the pattern for the Taylor polynomials of $\sqrt{x}$ about a point a. (Alternatively, work out one from the group work.)

  Prove that this is the right pattern by induction.

  (Computing Taylor series requires good skills in pattern recognition.)

Group Work/Examples.

- Find the N^{th} Taylor polynomials of $f(x) = \ln(1 - x)$ by developing the pattern in the derivatives. Compare with representations found earlier.

  Prove the pattern is right by induction.

  Could do the same for $f(x) = x^{-\frac{1}{3}}$ about $x = 1$.

  $f(x) = x^{-\frac{1}{2}}$ about $x = 1$ worked well, too. It leads to $T_1(x) = \displaystyle\sum_{n=0}^{N} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}(x-1)^n$.

  Advantage over $\sqrt{x}$ is that there are no terms that need to be split off. Pattern is better to recognize. Computer picture can be used to lead into the need for error analysis.

- Activity on p. ACT-36.

- Find the N^{th} McLaurin polynomial of $f(x) = x^3 + x^2 + x + 1$ about $a = 1$. Would you expect this to be an infinite series?

- Find the third Maclaurin polynomial of $\dfrac{e^x}{1-x}$.

Section 16.1.3. Error Estimates for Taylor Polynomials

Suggested Time. 1 class.

Lecture/Presentation.

- Taylor series need not converge (everywhere) to the function. Could use $f(x) = x^{\frac{1}{3}}$ from instructor's guide for Taylor polynomials or $f(x) = \sqrt{x}$ as examples.
- Use of Taylor polynomials in the design of operating systems and in the computation of look-up tables (which might in turn be part of operating systems). I'm not sure how much of the on-line computation via Taylor polynomials is history by now and replaced by other algorithms or lookup tables. The Commodore 64 did use Taylor polynomials, but those were the 1980s.
- Present Theorem 16.1.11. Explain the role of M.
- For the function $f(x) = e^{-\frac{x}{2}}$ and $N = 7$ find the error of the approximation with the Taylor polynomial on $[-5, 5]$.
- For the function $f(x) = e^{-\frac{x}{2}}$ find N so that $|f(x) - T_N(x)| < 10^{-8}$ for $-10 \le x \le 10$.

Group Work/Examples.

- For the function $f(x) = e^x$ and $N = 10$ find the error of the approximation with the Taylor polynomial on $[-3, 3]$.
- For $f(x) = e^{-\frac{x}{2}}$ find the second Maclaurin polynomial and estimate the error in the approximation on $\left[-\frac{1}{2}, \frac{1}{2}\right)$ and on $[-2, 2]$.
- For the function $f(x) = x^{\frac{1}{3}}$ expanded about $a = 2$ find find N so that $|f(x) - T_N(x)| < 10^{-3}$ for $1 \le x \le 3$.

  Other good candidates: $f(x) = \sin(x)$; $f(x) = \dfrac{1}{1-x}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$; etc.
- Activity on page ACT-37.
- Suppose we know that $f(0) = 1$, $f'(0) = 3$, $f''(0) = -2$, $f'''(0) = 0$, $f^{(4)}(0) = 9$ and that $|f(5)(x)| \le 5$ for all x in $[-5, 5]$. How close is the fourth order Maclaurin polynomial of f to f on the interval $[-5, 5]$?

Reading Quiz.

- Working with Taylor polynomials, functions are approximated by

  ☐ Sums of powers of x,

  ☐ Sums of trigonometric functions,

  ☐ Sums of real exponential functions,

  ☐ None of the above.
- The general N^{th} Taylor polynomials of a function can

☐ Never be computed,

☐ Always be computed,

☐ Be computed only if we can find a general pattern for $f^{(n)}(x)$,

☐ Not necessarily be computed, but we can compute Taylor polynomials of arbitrary high degree.

Section 16.2. Numerical Integration

Suggested Time. 1-2 class periods. Could break it up into the methods one day and the error estimates the other day. Students can find the error estimates surprisingly challenging.

Lecture/Presentation.

- May want to know an integral of a function that has no nice antiderivative or is only given as data. (Example of time dependent data.)
- Go back to the geometric definition of the integral. Show left and right sided sums, implement right sided sum in MathCAD. Template inputs f, a, b, n.

  Use $f(x) = \sin(x)$ on $\left[0, \frac{\pi}{2}\right]$ as sample function.
- Show the idea behind the more sophisticated rules geometrically, re-emphasize the idea of cutting things up and then summing.
- Show formulas for the midpoint, trapezoidal and Simpson's rule (derivation of Simpson's rule did not have the desired effect when I did it; possibly better to drop it in favor of more coverage on errors)
- The two ways in which one encounters data. (This is the underlying idea to numerics as well as statistics.)

  1. **A posteriori or after the fact.** This is the data we generated. How good is it?
  2. **A priori or before the fact.** Here is what we want to measure. How do we set it up to be most efficient?
- Show formulas for error bounds.

  Explain the mysterious constant coming from the appropriate derivative very carefully. Emphasize the constant is the maximum of the absolute value of the derivative.

  Essentially we approximate with functions that can only bend so much. The derivative is a measure of how far the actual function can bend and large values there will give us large errors.

Group Work/Examples.

- Question: If we did not know the exact value of the integral (and we usually don't), how do we know how good the number we computed is?
- How large do we need to choose n to approximate $\displaystyle\int_0^2 \sin(x^2)dx$ to within 10^{-4} using the midpoint rule?

  Derivatives can be computed with CAS. This is an a priori estimate. (A posteriori is the direct computation of the error.)
- How large would we need to choose n above when using Simpson's rule?
- Why do the error bounds show that Simpson's Rule is likely to be most efficient?

- Activity on page ACT-35.

Reading Quiz.

- In deriving Simpson's rule the area under a curve is approximated using
  - ☐ Rectangles.
  - ☐ Trapezes.
  - ☐ Regions with three straight and one parabolic boundary.
  - ☐ None of the above.
- In which situation below do we NOT need to use approximate integration to determine the definite integral.
  - ☐ We know the values at certain values for x (function is given as data)
  - ☐ We know the function's antiderivative.
  - ☐ We know the function's formula f(x)=...
  - ☐ We know the formula f(x)=..., but there is no nice formula for the antiderivative
- What does the mysterious K or C in the error bound formulas stand for?
  - ☐ I was asking myself that very same question.
  - ☐ It is a fixed constant of nature.
  - ☐ It is an upper bound for the absolute value of the appropriate derivative.
  - ☐ It is an upper bound for the second derivative.
- Using the trapezoidal rule on a function with bounded second derivative and a fixed interval, the error made in the approximation
  - ☐ Can be made arbitrarily small.
  - ☐ Will be zero for $n > 1000$.
  - ☐ Ensures that digits past the fourth digit are always wrong.
  - ☐ The trapezoidal rule always gives the exact answer.
- Which of the approximate integration methods is usually the most accurate. That is, which one will normally give the best approximation with the fewest evaluations of the function?
  - ☐ Midpoint Rule
  - ☐ Trapezoidal Rule
  - ☐ Simpson's Rule
  - ☐ They are all about equal

Notable Homework Problems.

- Project 16.4.1 can be assigned after this class period.

Section 16.3: Newton's Method.

Suggested Time. 1 class period

Lecture/Presentation.

- Possible lead in: why do some calculators and CASs need an initial guess for their root finder?
- Introduce Newton's method graphically by computing the zero of the tangent line.
- Re-emphasize the idea of recursion. We know we make a mistake and by repeatedly correcting it we get the right result.
- Show example/animation of Newton's method started with $x_0 = 2$ for the function $f(x) = x^3 - 5x + 5$. This is similar to Example 16.3.3 with $x_0 = 1$, but the demonstration should be helpful. Could also use Example 16.3.3 itself.
- Emphasize that Newton's method can be used to solve any equations, so if this comes up in, say, finding the zeroes of a derivative, still the same principles apply.

Group Work/Examples.

- Activity on page ACT-28
- (Can ask for a variety of things from the following.) Use Newton's method to find the zeroes of the function. If an initial value is given, use that value. If not, find all zeroes. (4 digits behind the decimal point should suffice for the approximation.)

  Or: Find the first few values of the sequence generated by Newton's method.

  Possible add-ons: Also determine after how many iterations your calculator output does not change any more. Check if your output really is a solution by plugging it back into the function.

  1. $f(x) = \cos(x)$, $x_0 = 1$
  2. $f(x) = x^6 - 5x - 12$
  3. $f(x) = x^3 - 3x + 4$, $x_0 = 1$ (has horizontal tangent at the starting point).

- Use Newton's method to solve the equation (similar add-ons as above possible)

  1. $\cos(x) = x$
  2. $\tan(x) = 1 - x, 0 \le x \le \dfrac{\pi}{2}$

- Use Newton's method to find $\sqrt[3]{2}$ correct to four places after the decimal.
- Find the critical points of $f(x) = x\cos(x)$ in $\left[0, \dfrac{\pi}{2}\right]$.

Reading Quiz.

Notable Homework Problems.

- Exercise 8 is an (albeit construed) example how large numbers and scales can affect numerical results.

Scoring Rubric for Project 16.4.1

The scoring rubric below is approximately what will be used to score the numerical integration project. It can be used as a check sheet to determine if all points that need to be made were made. While completeness according to this rubric does not guarantee a perfect score, incompleteness certainly guarantees an imperfect score.

Assessment item	percentage	check/score
Inputs encoded correctly	10%	
Correct derivative for chosen method	10%	
Automatic computation of K	10%	
Correct formula for n	10%	
Graph of the appropriate derivative to double check K	10%	
Correct summation formula	10%	
Integral of the function with built-in numerical processor to double-check	10%	
Absolute value $\|N - C\|$	10%	
Correct solutions for the tasks	15%	
Template works without problem for any new set of inputs	5%	

Section 17.1. Fundamentals on Probability Functions.

Suggested Time.
1 class period.

Lecture/Presentation.

- First step in analyzing randomness: Develop a mathematical framework and vocabulary.
- Sample space: set of all possible outcomes.

  Name sample spaces for (repeated) coin flips, randomly pulling cards from a deck, roulette, etc.
- Event: subset of a sample space.

  Examples in coin flips, maybe display a roulette table layout and explain some events.

  `http://www.roulette.sh/table_layout.html`

  Record the events.
- Explain mutually exclusive/disjoint events and complementary events. Use examples collected so far.
- Explain properties of a probability function, use discrete uniform probability function as an example.
- Draw histogram for fair and loaded dice, explain the mass analogy through the fact that added probability in one place will bias the process towards that outcome, just as added mass in one place changes the center of mass.
- Define a discrete random variable as a real valued function on a sample space.
- Explain the Bernoulli distribution.

  Show distributions for coin flip and for betting red in roulette.
- Define expected value.

  Compute the expected value for coin flip and betting red in roulette.

Group Work/Examples.

- Reminder of processes influenced by randomness: Let students all flip a coin and find by show of hands how many had "heads" and how many had "tails".
- Then let students give examples of daily (engineering) life processes as possible that have a random component.

  Steer the discussion to the realization that measurements are influenced by random effects and that manufacturing processes do not produce the same object every time.
- Do the named processes all have the same behavior? No. They give different (types of) values, some cluster (measurements), some don't (coins).

Section 17.2. Continuous random variables.

Suggested Time. 1 class period.

Lecture/Presentation.

- Show EXCEL's RAND() function to demonstrate what a continuous random variable can be.
- The phenomena we will concentrate on in this module are phenomena that can return values in a continuum such as the reals or subintervals of the reals.
- Histogram visually depict the likelihood a value occurs in a certain interval. The likelihood is proportional to the area above the interval.
- Use the tool `histogram to density function.xls` to show how refinement of the histogram (under the assumption of enough measurements available) approaches the shape of the density. Show how the re-scaling of the heights from $P([a,b))$ to $\dfrac{P([a,b))}{b-a}$ gives the density function ("Density = probability per interval length").
- Events are sets (in this text mostly intervals). This could be interpreted as saying that the random experiment returned a value within a certain set.
- Present the properties of a probability density function.
- Note that continuous random variables are defined via their density functions.

  We are taking a shortcut here and don't worry about assigning probabilities to individual outcomes in the sample space any more. (In fact, this is not possible, and we are not going into measure theory in this text.)
- Emphasize the interpretation of area as probability.
- Some examples of probability density functions and functions that are not density functions.
- Brief discussion of the usefulness of continuous models.

Group Work/Examples.

- Which of the following phenomena is random? The weight of a candy bar purchased from a candy machine or the change received for the dollar you inserted?
- Ask students to determine which histograms in Figure 17.6 come from the same process.
- Which of the following is a continuous random variable? The score of a roll of a die? (not continuous) The actual weight of the purchase when buying sliced ham at a grocery store? (crv) The price of three cans of soda, given that one can costs 50 cents? (deterministic) The actual fill height of each of these cans? (crv)
- Determine which of the following functions are probability density functions. For those that are not probability density functions, attempt to rescale them so that they become probability density functions.

- $f_X(x) = \begin{cases} \frac{4}{x^5}; & \text{for } x \geq 1, \\ 0; & \text{otherwise.} \end{cases}$
- $f_X(x) = \begin{cases} \frac{4}{x^5}; & \text{for } x \geq 5, \\ 0; & \text{otherwise.} \end{cases}$

  (rescale)
- $f_X(x) = \begin{cases} \frac{1}{x}; & \text{for } x \geq 1, \\ 0; & \text{otherwise.} \end{cases}$
- $f_X(x) = \begin{cases} x; & \text{for } -1 \leq x \leq 1, \\ 1; & \text{for } 1 < x \leq 2, \\ 0; & \text{otherwise.} \end{cases}$

- Compute some probabilities with the densities above.

Section 17.3: Some Widely Used Density Functions

Suggested Time. 2 class periods: One on exponential distributions, one on normal distributions.

Lecture/Presentation.

- Quick example on the uniform distribution. Note that that `rand()` function in most spreadsheets and computer languages is uniformly distributed on [0, 1].
- Introduce the exponential densities, explain how the parameter affects the distribution of the probability.

  Note that exponential distributions are typical "waiting time distributions".

  Give examples of computing a probability and finding the parameter.
- Introduce the normal distribution densities, explain the effects of the parameters.

  Note the central role of normal distributions for measurement processes and through the central limit theorem.

  Give an example of computing a probability. Note how we need numerical tools to do this. Use this as a lead in to the theorems that put the standard normal distribution front and center.
- Prove the theorems on the normal distribution to show how we transform to the standard normal distribution. When solving problems, show immediately how we transform back to Z.

  Note especially that for normal random variables the probabilities of intervals $[\mu - z\sigma, \mu + z\sigma]$ are independent of the actual values of μ, σ.
- Find an upper bound for $e^{-\frac{x^2}{2}}$ for large x, say $e^{-|x|}$ to help with problems **??** and 13.

Group Work/Examples.

1. The waiting time for customer support at WGTYOMONEY.com is exponentially distributed with $\Theta = 4min$. What is the probability to wait less that $1min$?
2. The waiting time at a QuickLube shop is exponentially distributed. If the probability to wait longer than 20 minutes is 10%,
   - Find Θ,
   - Find the probability that the waiting time is less than 10 minutes.
3. The hourly output of a machine that makes paper clips is normally distributed with $\mu = 100lb$ and $\sigma = 5lb$. Find the probability that the machine produces less than $90lb$ of paperclips in a given hour.
4. The hardness of steel rods produced by a company is normally distributed with $\sigma = 2$ (Rockwell 2 units). How should μ be chosen if we wish to have 95% probability that each steel rod has at least a hardness of 61 Rockwell 2 units?

5. For a random variable X find an interval that contains 3 such that we have a $\geq 90\%$ probability that X returns values in the interval. Assume X is

   (a) Uniformly distributed on [0, 10],

   (b) Exponentially distributed with $\Theta = 1$,

   (c) Normally distributed with $\mu = 3$ and $\sigma = 1$.

   (This could be broken up. Every group works one part and then reports.)

6. Why do we want to work with the standard normal distribution?

Section 17.4. Cumulative Distribution Functions

Suggested Time. $\frac{1}{2}$-1 class period. Computer lab with spreadsheets needed.

Lecture/Presentation.

- Compute the distribution function for the exponential distribution (easy substitution exercise).
- Connection between densities and cumulative distribution functions through the fundamental theorem of calculus.
- The properties of cumulative distribution functions.
- Need to work with cumulative distribution functions because for many distribution functions the integrals cannot be solved symbolically.
- Compute some probabilities for standard normal distributions using the table of the cumulative distribution function.
- Availability of cumulative distribution functions on spreadsheets and computer algebra systems.
- For a graphical implementation of the cumulative distribution function of the standard normal distribution consider
  `http://davidmlane.com/hyperstat/z_table.html`

Group Work/Examples.

- If no spreadsheet is available, one could use the enclosed activity on the χ^2-distribution on page **??** to get students started on working with tables.
- In what other situations did we assign a name for a complicated procedure and then worked with the new function rather than worrying about how it is computed? (Square roots, logarithms, exponential functions for real exponents.)
- Can a cumulative distribution function be constant on some interval?

Reading Quiz.

1. Hand in your answer for question 5 in Activity **??**
2. Please draw or give numerically a density function for which the cumulative distribution function has the values
   - $c(0) = 0$
   - $c(1) = .3$
   - $c(2) = .5$
   - $c(4) = .6$
   - $c(5) = .9$
   - $c(10) = 1$
3. The function on the right **(draw one)** could be

(a) A probability density function.

(b) A cumulative distribution function.

(c) Both.

(d) Neither.

4. For a standard normal random variable X compute

- $P(X < .8) \approx$
- $P(X > 1.6) \approx$
- $P(.2 < X < 2.6) \approx$
- $P(X < -.6) \approx$
- $P(-1 < X < 2) \approx$
- Find an approximate value for t such that $P(X < t)$ is .75
- Find an approximate value for t such that $P(X < t)$ is .18
- $P(X \text{ not in } [1, 2]) \approx$

Some values of the cumulative distribution function
$F_X(t) := \int_0^t \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{\frac{-x}{2}} dx$ **of the** χ^2**-distribution with** $n = 2$

t	$F_X(t)$ (approx.)	Examples/Exercises
0.0	0.0000	
0.2	0.0952	
0.4	0.1813	$P(2.8 < X < 4.2) \approx 0.8775 - 0.7534 = 0.1241$
0.6	0.2592	
0.8	0.3297	
1.0	0.3935	$P(X < .8) \approx$
1.2	0.4512	
1.4	0.5034	
1.6	0.5507	$P(X > 1.6) \approx$
1.8	0.5934	
2.0	0.6321	
2.2	0.6671	$P(.2 < X < 2.6) \approx$
2.4	0.6988	
2.6	0.7275	
2.8	0.7534	Find an approximate value for t such that
3.0	0.7769	$P(X < t)$ is .75
3.2	0.7981	
3.4	0.8173	
3.6	0.8347	Find an approximate value for t such that
3.8	0.8504	$P(X > t)$ is .18
4.0	0.8647	
4.2	0.8775	
4.4	0.8892	$P(X$ not in $[2, 4]) \approx$
4.6	0.8997	
4.8	0.9093	
5.0	0.9179	
5.2	0.9257	
5.4	0.9328	
5.6	0.9392	
5.8	0.9450	
6.0	0.9502	

Section 17.5. Mean and Variance.

Suggested Time. 1 class period.

Lecture/Presentation.

- Introduce the definition of the mean and the variance.
- Mean and variance are measures of location of center and spread around the center.
- Connection to mechanics through the center of mass of a linear density or the x-coordinate of the center of mass of a lamina: Explain the analogy between center-of-mass and expected value formulas.
- Prove linearity of $E(\cdot)$ and $V(X) = E\left(X^2\right) - (E(X))^2$.
- Compute mean and variance of the uniform and exponential distributions.
- Visualize mean and variance in pictures of the distributions we have.
- It is possible to choose mean or variance in such a way that certain events have certain probabilities.

Group Work/Examples.

- When trying to find the center, which values/intervals should be given the most "weight"?
- Students compute mean and variance of the exponential distribution.
- Activity "Mean and Variance of a Normally Distributed Random Variable" on page ACT-39,

Section 17.6: Questions to ask when solving statistics problems

Suggested Time. 1 class period.

Lecture/Presentation.

- **Central Theme.** The equation/inequalities $P(a < X < b) <=> \alpha$. Much of what we do in these notes revolves around the correct setup and interpretation of equations/inequalities of this type.

Group Work/Examples.

- Activity "Choosing the mean to achieve a certain probability" on page ACT-40.

Section 18.1. Sample Statistics.

Suggested Time.

$\frac{1}{2}$ class period.

Lecture/Presentation. My best idea right now is to just follow the text.

- To estimate parameters of given random distributions we must use functions that only access measurable data.
- Exact form of sample variance seems only justifiable through the theory. It can however be made plausible with "soft" arguments as in the text.
- Use Figure 17.7 to illustrate that we are now moving back up in the indicated hierarchy to use sample statistics for making inferences about the underlying distribution or population.

Group Work/Examples.

- Find examples of sample statistics and of quantities that are not sample statistics. For example $\sum_{k=1}^{n}(x_k - \mu)$ (no), $\sum_{k=1}^{n}(\cos(x_k) + \sin(x_{n-k})$ (yes),

Sections 18.2. Statistical behavior of the sample mean.

Suggested Time.
1-2 class periods, depending on how much of the section is covered. Central limit theorem can take a day.

Lecture/Presentation.

- Formulate the Central Limit Theorem.
- The central limit theorem only applies to populations of means, not to populations of *individual* samples.
- The approximation of the normal distribution is bad for means of small samples.
- Solve a problem similar to the following. The mean value of the maximum nondamaging downward force that a soda can from a given company can withstand is 240 pounds with a standard deviation of 5 pounds.
  - What is the probability that in a batch of 200 cans the average maximum nondamaging downward force is at least 235 pounds?
  - Find an interval such that with 95% probability the average maximum nondamaging downward force for a batch of 200 is in the interval.
  - Find the interval as indicated that has the highest starting point.
- For small samples use the t-distribution.
- Theorem 18.2.8 and its applications.
- Show graphically how the t-distribution approximates the normal distribution for the degree of freedom getting large. (Mathcad tool)
- Central limit theorem as well as Theorem 18.2.8 are applied in the same way.
- The usefulness of the theorems lies in the ability to gauge the quality of data gathered and to design experiments so that the data gathered will have a certain quality.

Group Work/Examples. Activity "Using the Central Limit Theorem" on page ACT-42.

- What do we do if the standard deviation is not known?
- Why do we not use the central limit theorem for all computations?
- Does the hypothesis of Theorem 18.2.8 always hold?

Reading Quiz.

1. Which of the following statements is a correct verbalization of the central limit theorem?

   (a) Every sufficiently large sample population is approximately normally distributed.

   (b) Given sufficiently large individual sample size, random sample averages are approximately normally distributed.
   (c) Every sample population is approximately normally distributed.
   (d) Sample averages are always approximately normally distributed.

2. When taking measurements what will need to be done to cut the standard deviation of the sample averages in half?
   (a) Cut the sample size in half.
   (b) Double the sample size.
   (c) Quadruple the sample size
   (d) Increase the sample size tenfold.

3. The central limit theorem applies
   (a) Only when the sampled population is normally distributed.
   (b) To any sampled population.
   (c) Only for sufficiently large samples.
   (d) For all sample sizes.

Section 18.3. Confidence Intervals.

Suggested Time.
1 class period.

Lecture/Presentation.

- Define z_α in the context of confidence intervals. Give an example how to compute z_α.
- Reported data will always have an error attached to it, so it is important to have range and an idea how confident one is about the actual value being in that range.
- Derivation of the confidence intervals presented in Section 18.3.
- Note the importance of the normality assumption for small samples and for estimation of the standard deviation.
- 50 water samples from a lake have been measured for their PCB (PolyChlorineBiphenals) content. The average content was $7.3\frac{\mu g}{l}$ with a standard deviation of $0.2\frac{\mu g}{l}$.

  Find a 90% confidence interval for the lake's PCB content per liter.
- How many samples should be taken to obtain a 99% confidence interval that has length $0.02\frac{g}{l}$?

Group Work/Examples.

- Sheet "Two ways to compute the error" on page **??**. Emphasize that either way to compute the function of v and its error is correct and that as long as we stay away from bad singularities it is much more important to understand the source of the error than the details of the two mechanisms of propagation. Activity "Designing Confidence Intervals" on page ACT-45.
- Why do we want our confidence interval to be symmetric? (Best idea in the absence of any other constraints. In hypothesis testing we will see that things can get shifted, for example for testing $\mu < \mu_0$.

Section 19.1. Series of Numbers

Suggested Time. $\frac{1}{2}$ class

Lecture/Presentation.

- Infinite sums can have a finite or an infinite value,
- Historically the problem was that people had a hard time understanding that a "sum with infinitely many terms" can have a finite value. Interestingly enough, students who study infinite series for the first time can become so ingrained with the insight that the sum can be finite that they automatically assume the sum is finite whenever the limit of the terms is zero.

  This can be due to an ill-fated hope that the converse of the limit test is true. Careful discussion of the harmonic series is intended to help here.

  To keep prerequisites minimal, the integral test is not used in the discussion. It can be used in classroom presentations if integrals are available. It's also Exercise **??** in Section 19.2.
- Motivate series by considering a patient that is injected a certain dosage of medication every day. Assume that half the medication the patient has in the body is broken down over 24 hours. Question: Will this patient o.d.? (This is the activity on page ACT-2.)
- Show geometrically how infinitely many pieces can sum to a finite number. This is most natural for $\sum_{n=1}^{\infty} \frac{1}{2^n}$, but other geometric series are possible, too. The complement of the ternary Cantor set would be too abstract, though.
- Define convergence of a series.
- Convert an infinite decimal into a fraction,
- Show how to re-index sums, possibly in the context of converting an infinite decimal back into a fraction,
- Limit test as a first test for divergence, harmonic series as an example that shows the limit test is not an equivalent criterion,

Group Work/Examples.

- Prove the formula for the partial sums of the geometric series with induction,
- Activity on page ACT-2.
- Determine which of the following series is convergent. For those that are convergent, determine the limit if possible.

  1. $\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$
  2. $\sum_{n=1}^{\infty} \frac{n}{n+1000}$
  3. $\sum_{n=1}^{\infty} \frac{500}{4^n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$ (foreshadows comparison test)

5. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ (foreshadows comparison test)

It could also be useful to let students compute the first few partial sums of each series to demonstrate that series that start out fast can still converge and series that start out slow can still diverge.

Reading Quiz.

- $\displaystyle\sum_{n=1}^{4} 3n =$
  - ☐ 30
  - ☐ 3
  - ☐ $\left.\dfrac{3n^2}{2}\right|_1^4$
  - ☐ $12n$
- The sum $\displaystyle\sum_{n=s}^{\infty} a_n$ of an infinite series is
  - ☐ What we get when we sum infinitely many numbers.
  - ☐ The limit of the terms a_k.
  - ☐ The limit of the sequence of the partial sums $\left\{\displaystyle\sum_{n=s}^{k} a_n\right\}_{k=s}^{\infty}$ if it exists.
  - ☐ Always infinite.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
  - ☐ Does not converge.
  - ☐ $= 1$.
  - ☐ $= 0$.
  - ☐ Converges by the monotone sequence theorem.

Notable Homework Problems.

- Problem 6 is the classical "Achilles and the tortoise paradox" that exhibits where ancient greek mathematics got "stuck".
- Problem 9 is very technical, as it requires estimation skills that are quite specialized. Still, in a class with many mathematics majors, this may be a beneficial preparation for heavier emphasis on approximating integrals using power series solutions.

Section 19.2. Some Convergence Tests for Series

Suggested Time. 1 class period

Lecture/Presentation.

- Prove some of the tests in the section: comparison test, p-series test, absolute convergence implies convergence, ratio test.
- If a series is bounded by a series that does not converge, we cannot conclude anything. Possible examples: The harmonic series as an upper bound of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ and of the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- Absolute convergence implies convergence but not vice versa.
- The the geometric series and the comparison test are leading to the ratio test.
- The ratio test is important in the investigation of power series.
- If a series is given with no hint which test to use, then the determination of convergence or divergence is a process similar to integration.

Group Work/Examples.

- Activity on page ACT-46.
- Show that the series converges or diverges using the ratio test. If the ratio test is inconclusive, try to use another test.

  1. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 4^n$
  2. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$
  3. $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$
  4. $\sum_{n=0}^{\infty} \frac{n^2}{n^5+8}$
  5. $\sum_{n=0}^{\infty} \frac{n!}{10^n}$

Reading Quiz.

- The series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

  ☐ Converges by comparison test and p-series test
  ☐ Converges by ratio test
  ☐ Diverges by ratio test
  ☐ Diverges by comparison test and p-series test

- The series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$

  ☐ Converges by comparison test and p-series test
  ☐ Converges by ratio test
  ☐ Diverges by ratio test
  ☐ Diverges by comparison test and p-series test

Notable Homework Problems.

- Exercises 7 (alternating series test), **??** (integral test), and 9 (limit comparison test) present a few more convergence tests for series. These could be discussed in class if so desired.

  Exercises 8, 5, 10 and 11 provide more practice with these tests.

- In Exercise 5 we prove the p-series test using the integral test.

- Exercise 12 presents the numerical estimation of series.

Section 19.3. Power Series.

Suggested Time. 1 class period.

Lecture/Presentation.

- Motivate the idea of a power series through the observed convergence of Taylor Polynomials.
- Define power series, state the theorem about the radius of convergence. If available, show with previous examples from Taylor series (say the series for $\sqrt{x}$ about $x = 1$, which is $1 + \frac{1}{2}(x-1) + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2n-3)!}{2^{2n-2}n!(n-2)!}(x-1)^n$) that convergence need not happen everywhere.
- Define series as a first step towards understanding power series
- State the ratio test
- State theorem on derivatives of power series
- The typical math text motivation for representing all kinds of functions using variations on the pattern $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is the approximation of integrals with series. This looks thin to me, because in any application I can imagine, people would use numerical integration (after all, they are out to get a number).

  Yet this representation of functions is beautiful, and I believe beneficial, pattern recognition.

  In electrical engineering the Z-transform depends on the above recognitions and translations. I plan to put something on the Z-transform into the text. (Just need to find the time.)

Group Work/Examples.

- Determine if the series converges (review)

  1. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$
  2. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$
  3. $\sum_{n=1}^{\infty} \frac{n^2}{n+1000}$
  4. $\sum_{n=1}^{\infty} \frac{1}{n}$ (intentional, test fails)

- Compute the radius of convergence of the power series.

  1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^n x^n}{n!}$

- Find the derivative of the power series (lead in to series solutions of DEs)

  1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
  2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ (find the second derivative)

- Determine if the given series solves the given differential equation.

  – $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$, $y'' + 4y = 0$

Reading Quiz.

Notable Homework Problems.

- Exercise 5 is an example in which the error formula for Taylor polynomials only proves convergence for half the values within the interval of convergence.
- Exercise 6 provides the series expansion of $(1 + x)^k$ for arbitrary k. This is used in some applications, but in the end it's a formula that I would look up.
- Project 19.5.1 can be assigned after this section.

Section 19.4. Multiplying and Dividing (Power) Series

Suggested Time. $\leq \frac{1}{2}$ class. This section could also be omitted or delayed until reduction of order for series solutions is covered.

Lecture/Presentation.

- Power series multiply and divide "essentially like polynomials".
- Find the product of $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty}(n+1)x^n$. (Can be verified by comparing with the expansion of $\frac{1}{(1-x)^3}$, which we could obtain by division or by twice differentiating $\frac{1}{1-x}$.)
- Find the first few coefficients of the quotient of $3+x$ and $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$. (Can be verified later using $(x+3)e^{-x}$).

Group Work/Examples.

- More computations as necessary or desired.

Reading Quiz.

Notable Homework Problems.

- In Exercise 6 we prove Pythagoras's Theorem via series.