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Preface

The foundation of mathematics is not found in a single discipline, because mathematics is a general way of thinking in a very rigorous logical fashion. This text is written especially for students who are about to make first contact with this way of thinking.[1] No matter what the first "proof class" in your curriculum is, you may have heard stories from upperclassmen about how hard it is. The bad news: They are right. Proofs are hard. The good news: You can adjust to this new way of thinking, and it will make you more capable in general.

The biggest challenge in a first proof class is that you need a certain mental discipline to not use mathematics that you already know (all that calculus you took) to prove some rather simple-looking results. In fact, for some early theorems, it is very common to ask yourself "Why do we even need to prove this?" or to realize "I already know this, but why is it true?" Check out the questions on page x for some simple and hard questions that we will answer in this text.

To keep the temptation to use "known stuff" to a minimum, it is appropriate to start with nothing but formal logic and the axioms of set theory. But we must assure that we do not get too tangled in the technical details of these subjects. So the trick to this introduction (hopefully) is to give you enough of everything, but not too much of anything: Because mathematics is pure logical reasoning, the text is about logic. But formal logic is also a branch of mathematics, and this text is not a logic text. Because everything in mathematics is constructed from sets, the text is about set theory. But formal set theory is also a branch of mathematics, and this text is not a set theory text. Because mathematics deals with numbers, the text is about numbers. But formal number theory is also a branch of mathematics, and this text is not a number theory text. Because a lot of mathematics is algebraic, the text is about algebra. But formal algebra is also a branch of mathematics, and this text is not an algebra text.

So what is this text? Chapters 1-5, which I recommend for an introductory course, provide a rigorous, self-contained construction of the familiar number systems (natural numbers, integers, real and complex numbers) from the axioms of set theory. The construction in itself is quite beautiful, and it will train you in many of the proof techniques that mathematicians use almost subconsciously. But learning anything just for the sake of doing it is typically less efficient than learning something for a known purpose or within a known context (see [3]). Whenever possible, to increase motivation, I have included important applications of the fundamentals: Among other things, we

[1] But nonetheless, I hope that advanced students, and even experts, will enjoy it, too.

will discuss the scientific method in general (which is the reason why civilization has advanced to today's highly technological state), the fundamental building blocks of digital processors (which make computers work), and public key encryption (which makes internet commerce secure). To attach more familiar meanings to the entities with which we work in this text, and to specifically serve education majors, I have also included examples and exercises on a lot of the neat mathematics that we have learned in elementary and high school.

The various number systems were developed to better solve certain problems, most notably equations. So it is natural to cap the construction of the number systems with the solution of cubic and quartic equations in Section 5.8 and with the unsolvability of the quintic by radicals in Chapter 6. I don't expect Chapter 6 to fit into a standard one-term introductory course. But it makes for fascinating, if challenging, reading, and it could be used in a follow-up course or seminar.

Finally, Chapter 7 puts the finishing touches on our excursion into set theory. The axioms presented there do not directly impact our elementary construction of the number systems. But once they are needed in an advanced class, you will appreciate them.

Once you have gone through the majority of this text, I am confident that you will be prepared for your first "targeted" proof class, no matter if it is analysis, algebra, linear algebra (these are "the usual suspects") or another subject altogether.

That leaves us with the subject of proofs themselves. I have tried to present standard proof methods and to make the transition into proofs and abstract reasoning as smooth as possible. But that's the same as saying that a successful cross country running coach made the start of spring training as smooth as possible: The overall effort to go from untrained to competition level remains the same. All we can do is make the preparation as effective as possible. Moreover, unlike in, say, calculus, we will not be able to mimic examples. The only good example for a proof pretty much amounts to the proof itself. This is a radical shift for many of us, whether we are consciously aware of it or not. I had a hard time with it myself, because I did not even realize that the comparatively small number of techniques (examples if you will) in calculus had become second nature and sustained me through a lot of physics, for example.

To help us understand mathematical reasoning, reasons why proofs are set up in a certain way are frequently discussed in the early chapters. In fact, you should think about why a proof ran the way it did after *every* proof you read. Even if you don't find a good answer, the effort spent thinking about these ideas will help settle the necessary structures in your mind. (Consider [2] for the value of training that "feels hard".) In later chapters, proofs are presented in the way you would see them in a text for people who know how to do proofs. That means that as we progress, you will be required to fill in more and more gaps. This is quite customary in mathematics. If we were to fill in every detail of every proof, communication would be hard, because a lot of unnecessary detail would be included every time. In the video presentations (see [27]) many details are filled in verbally.

It is virtually certain that at some point you will wonder how your teacher or your classmates can produce proofs with seeming ease, while you cannot. This can be very frustrating, but just about everyone I know has gone through this struggle.[2] Think

[2]The only possible exception is an individual who started taking college classes at 12, graduated with a

of it this way: Do we really know what happens when we walk or pursue any daily activity that we may take for granted? Naturally, the task of walking is simple for a healthy, fully developed, non-inebriated adult.[3] But this ability must be acquired, too. Children spend hours, days, and weeks learning to stand up, stand without support, take one step, two steps and so on. Just because we can do it now does not mean it was always easy. So what exactly happens when we walk? Few people, and I am not one of them, even know all the muscle groups that have to work in very coordinated fashion as people walk. And the details of what happens in our bodies do not stop there. The muscle groups are steered by neural connections to the central nervous system, where a combination of conscious and unconscious processes governs how we walk. Conscious thoughts determine where we are going and at what speed, unconscious processes assure that we do not trip over our feet. A fully detailed description seems to be impossible. Moreover, an intellectual understanding of all the underlying processes will not make anyone, say, a better soccer player. On the other hand, a child who grew up playing the game can be a very good soccer player, even with no knowledge of physiology. So do proofs ("play the game") as much as you can, and chances are that they will become second nature.

Acknowledgements. Constructing the real numbers "from nearly nothing" is a natural start. My first analysis teacher, Professor Wegener, constructed $\mathbb{R}$ from $\mathbb{N}$ at the start of our class. At Tech, we used Clayton Dodge's out of print book [4], which constructs $\mathbb{R}$ in set theory, for the class for which I developed this text. Halmos' classic text [8] helped me with set theory, the Encyclopaedia of Mathematics [9] was an invaluable resource for quick references as well as some historical notes, my background in logic is from Hurd and Loeb [11], the idea to include a proof of the unsolvability of the quintic, and the proof itself, stem mostly from Maxfield and Maxfield [15], with Meyberg's texts [16] and [17] further helping with the presentation of Galois Theory, and [1] helping with Descartes' Rule of Signs. Finally, I used [9] and Wikipedia to refresh my memory on the few Latin words I know, on RSA encryption and as a general resource. The students in my Spring and Summer 2009 classes provided good feedback on the text and were patient with typos. Special thanks go to James Sims, who inspired Exercise 3-79, Wei Wang, whose ideas greatly improved Exercise 6-8 on Descartes' Rule of Signs, and Nichamon Naksinehaboon and Narate Taerat who corrected Exercise 6-44 on commuting cycles.

But most importantly, this work would not have been possible without the love and understanding of my family. They continue to live with me and my obsession for mathematics, and I am eternally grateful for that.

Ruston, Louisiana, August 26, 2009

– Bernd Schröder

BS in mathematics at 18 and had a Ph.D. in mathematics from a world-leading institution by age 23. He is an exceptional talent. Do not be disappointed if your progress is slower. Do not get discouraged when you get bogged down. That is simply normal.

[3] In case of inebriation, walking may be difficult, but it must be preferred over driving.

Questions

Human inquisitiveness is driven by questions. In high school, Einstein once asked himself a very simple question: If he had the appropriate means of transportation, could he catch up with the light emitted from a match he just lit? The answer he found about a decade later has revolutionized our understanding of the world. So it is good to ask questions, even questions that sound simple or strange. Of course, not all of us can ask and answer questions of Einstein's caliber. But answers to our questions can change *our* understanding of any subject, including mathematics. Think about the questions below, then start reading. References to the answers, if they exist as this text is published, are provided in parentheses.

1. Why is $1 + 1 = 2$? (Definition 3.7.)
2. Why is $3 < 9$? (Comment after Definition 3.20.)
3. Why can we not divide by zero? (Comments after Proposition 4.17 and after Proposition 5.1, as well as Proposition 5.11 itself.)
4. Is 3 a solution of $x^3 - 5x^2 + 3x + 9 = 0$? Why? Are there more solutions? If so, what are they? (Definition 4.55, Theorems 5.65, 5.66 and Exercise 5-63d.)
5. What is a set? (Sections 1.8 and 2.1.)
6. What is a digital computer made of? (Example 1.22 and Exercise 1-9.)
7. What is the scientific method? (Example 1.57.)
8. Why are publicly encrypted internet transactions safe? (Section 3.9.)
9. Why is there no quintic formula? (Chapter 6 – all of it!)
10. Why can angles not be trisected with straightedge and compass? (Exercises 6-5, 6-17, 6-32, 6-33 and 6-34.)
11. Can we count past infinity? (Section 7.2.)
12. Are there infinitely many twin prime numbers? (Unsolved, see Example 1.5 for the statement.)
13. Is there an efficient factorization algorithm for integers? (Unsolved, see the discussion after Theorem 3.107 for the relevance of this question.)