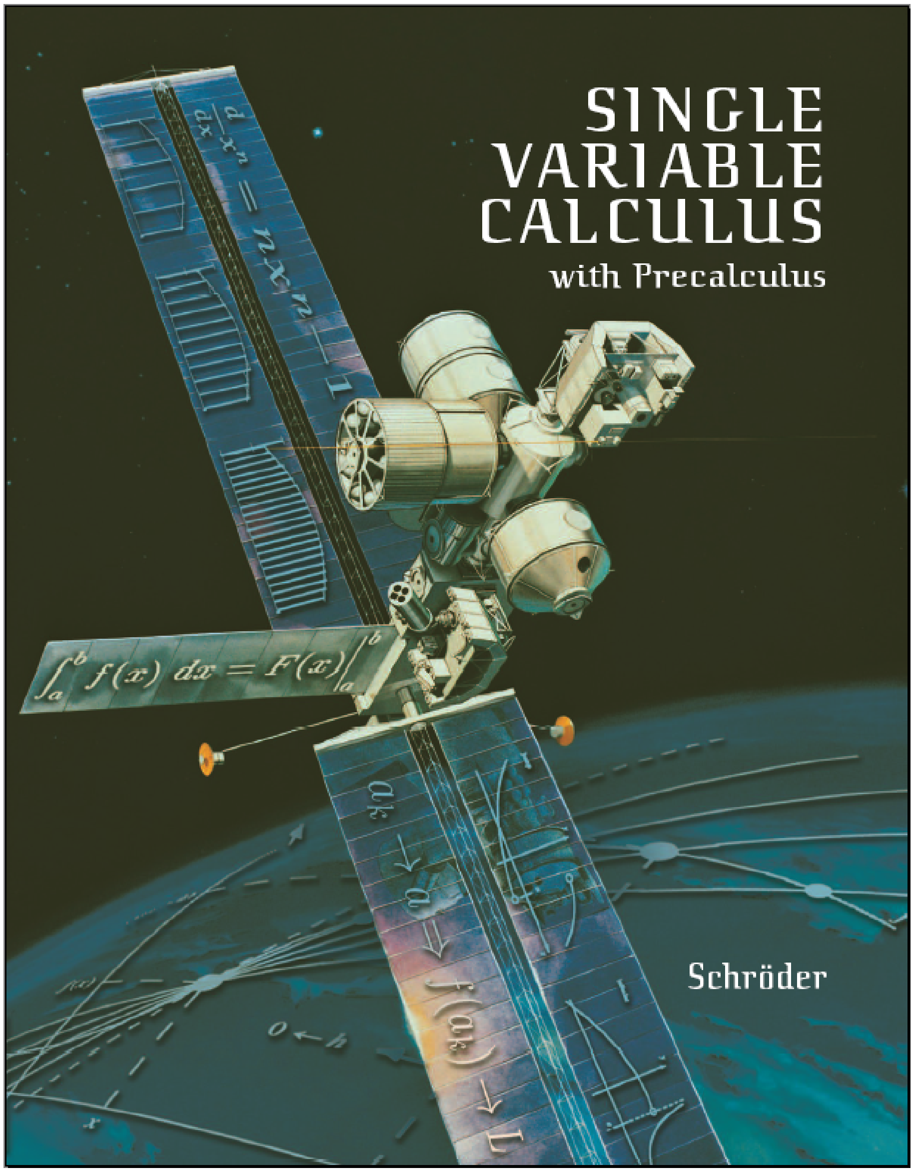

compiled: December 29, 2005

Single Variable Calculus

With Precalculus

Bernd S. W. Schröder

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Symbols In The Text

This text is the first half of a set of modules which covers freshman and sophomore mathematics as required by engineering, science and mathematics majors. Figure 1 shows the overall modular concept. References to results in modules that are not included are typed tiny and summarized in Appendix C starting on page 643. For example, **(DMV)** and DMV.A refer to explanations in this appendix. References to modules are in regular type. Sections, theorems, etc. are referred to in (parentheses).

Hints or solutions to selected exercises are given in Appendix D starting on page 647. These exercises are marked with numbers # on darker background.

For unrecognized terminology or symbols, please consider the index.

Foreword

This foreword presents ideas collected from colleagues, students and literature over the years. I cannot claim credit for any of them, because they all originated some time through conversations, observations, etc. I take responsibility for presenting and promoting them in the present form, though.

Feel free to take or leave any of the advice. Some statements may not apply to you, but consider everything at least once.

To The Student

Also read the part addressed to the teacher. The better you understand why material is presented in a certain way, the easier it will be to learn.

Congratulations! You made a good choice to study fundamental mathematics for science and engineering. Mathematics is ever present in these disciplines. Understanding it well will make it work as a tool for you, not be a cliff that detracts from the applications. Hence the mathematical foundation you build will be crucial to your further progress through your curriculum. To make your mathematical foundation as strong as possible, please consider the advice below.

The ultimate goal is to consistently function in mathematics at a high level.

Do not be intimidated. This overall project as outlined in Figure 1 takes you from precalculus to readiness for junior and senior level engineering and science classes. This goal was kept in mind throughout the development. The computational and conceptual challenge in assignments is designed to increase throughout the text. This is because the single variable methods presented here become parts of larger tasks in multivariable calculus. Thereafter, in applications even lengthy computations are dismissed as "just math," and that is not derogatory. The ultimate goal is to describe nature itself. Given the complexity of the goal, as soon as something is computable in principle, we have essentially solved the problem. Therefore you need to be ready to fill gaps in computations. Any price paid now solving challenging mathematics assignments will pay interest when you solve problems in applied classes. Lengthy computations will not unduly impress you and you will be able to focus on the science behind the computations.

Challenge yourself. The better the preparation, the stronger the performance, on tests as well as in future classes. Text and exercises were designed to allow you to challenge yourself. Homework problems vary from simple to challenging in every section. Examples are available for the fundamental tasks, but you are also expected to solve problems for which there is no example. The ultimate goal is to enable you to solve problems for which you have not seen an example or which even have never been solved before. In the same spirit, solutions in the back get sparser as you work through an exercise section. As a scientist or engineer, you must be able to function without an "oracle" that gives away the answer.

It's better to know something and be aware of how much else there is, than to know little and believe that is all there is.

If you ever get discouraged with slow progress, consider this. Slow progress usually means that a subject is challenging. Being aware of the challenge makes you more cautious and humble as you proceed. Humility and self esteem are two sides of the same coin. Self esteem makes you confident enough to start a task.

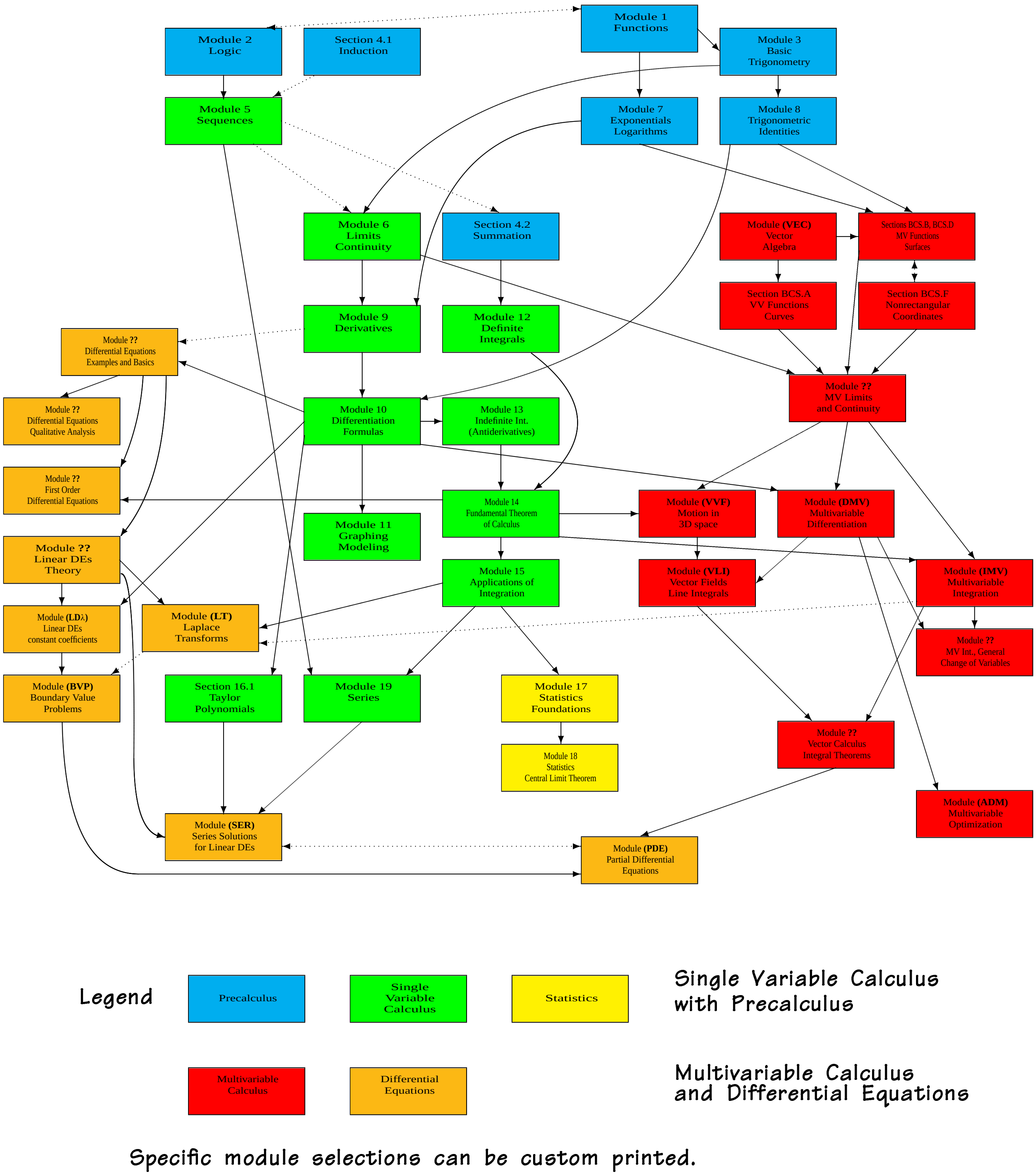

Figure 1: Flowchart of the logical dependence of the modular materials. Prerequisites are indicated by solid arrows. Dotted arrows indicate topics that reinforce each other or prerequisites that can possibly be omitted. The modules in this text are in white boxes.

Humility makes you cautious enough to check the results until the job is done right. Combine these two with the determination to succeed and you will go far. Your destination may surprise you, but getting there will be fun.

To read without a pencil is daydreaming.

Scientific reading means reading carefully and filling in gaps as necessary. Computations in technical texts often have gaps that the reader is expected to (be able to) fill. Being ready and able to routinely fill these gaps will help when you read advanced texts in your field. Computations that fill gaps are most effectively recorded in the margins of the text. The next time you read the passage, your note will allow you to proceed right away. Scientific reading is modeled throughout the text with handwritten notes in the margin. Sometimes the notes fill in a step in a computation. At other times, they remind you that you should fill in a step. When there are no notes, use your own judgment.

Notes in the margin will frequently encourage you to double check a result.

Check yourself frequently. Results should be double checked routinely. It is vital in any job that the overall result is correct, so you are acquiring another useful "habit of mind." Whenever possible, a double check should be independent of the computation. For example, instead of re-reading the solution of an equation, it is more effective and appropriate to substitute the result into the equation and check if it works. As you devise ways to double check your work you will also become more familiar with the content.

Structure your work and your thoughts. Mathematics is based on disciplined thinking and writing. Structured work is easier to read and double check. Consequently, you will be less likely to make mistakes. Moreover, by structuring your work on paper, you train your brain to impose structure on larger amounts of data. This mental structure improves your overall problem solving abilities. As soon as you have imposed a structure on a problem, it decomposes into a series of easier ones. You can then concentrate on these easier problems and you will know what to do next after each shorter step.

Make as many connections to your field and your interests as possible. As much as possible, I connected mathematics to content from parallel classes, or classes that you might take soon. I have also included applications that I find fascinating and stories around Dr. Smith to connect mathematics to what might pass for daily life. Explore and find connections to your field and your interests. The more you can connect the new content to your existing knowledge, the easier it will be to activate the mathematics when you need it.

Anything you learn is only useful if you can access it when it is needed.

Consider this example. In an experiment reported in [3], chess champions and non-chess-players were asked to reproduce "random" chess configurations from memory. Both groups performed equally as long as the configurations were meaningless. The chess champions' greater experience with the figures and the board, as well as their well developed ability to concentrate, did *not* provide any advantages. However, when the configurations looked random to the untrained eye, but could be meaningful in a game, the chess champions significantly outperformed the non-chess-players. To activate knowledge, the brain must form a connection between a new situation and stored information. This is what you do by connecting mathematics to other knowledge and experiences from your field or daily life. With these connections the transition from your field to mathematics and back is much easier than if mathematics is kept in a separate part of your brain.

Preference for one learning style does not prevent you from functioning in another. Any individual has capabilities in all learning styles, but usually prefers certain styles.

Learn about how you learn. Then use your strengths and work on eliminating or transforming your weak spots. Everyone has a unique learning style. This learning style influences how we prefer to study and what we prefer to study. You can explore your preferred learning styles at the learning styles inventory [39]. The site also provides advice how to function in a less preferred learning style.

Always emphasize your fundamentals. The use of fundamental mathematical tools is emphasized throughout. In some situations I have chosen to not present formulas that shorten a specific computation, because they might obscure a concept. The reason is simple. You will use your fundamental skills, chief of them algebra, throughout your educational and professional life (also cf. Figure 2). The more training you have in using them, the better. Therefore it is sometimes preferable to go through a slightly longer computation with fundamental techniques than to memorize a trick for a special case. The trick you may never need again, the fundamentals will never leave. If you had a tendency to memorize formulas, etc. to compensate for discomfort with fundamentals such as algebra, abandon your past approach. Too many students who work this way run out of gas.

The trick with learning is to not forget the things you need.

Repetition over time puts knowledge into long term memory. Many exercises in this text are designed to make you use results from another section. That means you need to remember the result, or, if memory does not serve, look it up and use it. The purpose of your studies is to become proficient in mathematics, not just in the specific section currently studied. Considering mathematics as a whole will serve you well, because you will see connections that cannot be seen with "one-section-a-day-tunnel-vision."

Learn to be ready to take a test every day at any time. The reason you learn is to have the knowledge available when it is needed. Since we don't know when that will be, it has to be *available all the time*. Once you reach that stage you are also ready to take a test any day, any time. Moreover, cramming before a test becomes a thing of the past. Being prepared does not mean to memorize every word in every book you read. It means, however, that certain fundamentals (definition of the derivative, differentiation rules, integration processes, trigonometric formulas, etc.) are *never* forgotten.

Increase your attention span. The short problems solved in early courses will become small parts of much larger problems. That means your attention span needs to increase, ultimately to hours. Although a human being's attention span is not infinite, when properly trained it will be up to the task. It's just a matter of what you are accustomed to. If you are used to classes that last 50 minutes, you can focus for that amount of time. A class that lasts 75 minutes will feel long in comparison, but after a while you will be able to focus that long. Your brain can and does adapt to the circumstances.

The key is training and to **avoid distractions**. Concentrated reading of texts and doing homework for extended periods of time is a crucial activity. This training should be paired with other measures. Sensory distractions (noise, music, don't even think about TV) should be avoided. Your brain needs to focus its resources on processing, not perception of unrelated effects. If no quiet place is available, try the library. Take breaks as necessary, but keep getting back to work for a set period of time. Then increase that time period. You will soon be able to stay mentally engaged for longer periods.

Distractions can affect your performance not just while training. Ever had a day where so many things happened early in the morning that you could not focus on anything? Too many perceptions put the brain in a state in which deep processing, as required for mathematics, science, engineering, is not possible any more. **Adequate rest** will allow you to start focusing again. Use down time for activities that recharge body and mind and get enough sleep. You can also minimize the overload. Is it really important to read that web site or watch that show? All perceptions during the day are processed in your brain. By avoiding some unnecessary ones you save "brain time" for more important tasks.

Above all, never give up. Mathematics and mathematical fields are challenging. Remember that human beings thrive on challenges and adversity. We would never have become a successful species by taking the easy way out. The challenges you face will keep you humble, so you will make sure you do the job right. Overcoming the challenges will give you the confidence to proceed to new ones.

To The Teacher

Also read the part that is addressed to the student. Several of the ideas explained there can stand some classroom reinforcement.

The calculus reform efforts starting in the late eighties and early nineties had many effects on undergraduate education. It was rightly pointed out that students need more conceptual understanding of mathematics. It was also rightly pointed out that we must not forsake fundamental, often computational, skills. I want to focus on the multiple benefits that can be derived from sound fundamentals paired with good conceptual understanding. Many of my remarks will seem obvious, because mathematicians usually combine the two with comparative ease. We should foster the same combination of skills in our students.

Solid fundamentals are mandatory in mathematics. It does not matter if these skills are computational and sometimes feel like drudgery. I often tell my students to not shy away from computation, because "once you know you can compute the solution of an applied problem, you have won, independent of how long or how hard the computation is." Moreover, our classes prepare students for classes in other fields such as engineering and science. Typical mathematical work in these areas consists of a lot of algebra with occasional conceptual spikes (cf. Figure 2). The conceptual spikes allow the creation of models. The subsequent algebra allows us to actually use the models to predict nature, design devices, etc. Our students will do a lot of computations in their careers and we might as well get them ready.

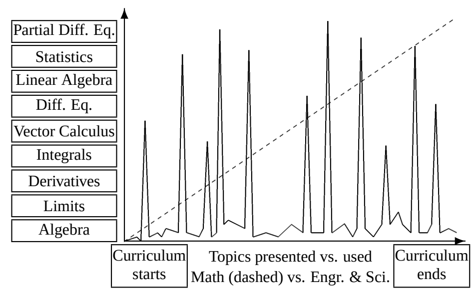

Figure 2: Comparison of the use of mathematics in other disciplines to the order in which it is usually presented. The horizontal axis represents time in a curriculum, the vertical axis represents mathematical topics in a typical order. The dashed line presents the progression through topics in mathematics, the solid graph represents the use of topics in the applied disciplines. In the applied disciplines much of the work is algebra, complemented by occasional conceptual "spikes" into more sophisticated topics such as integration, vector calculus, statistics or (partial) differential equations.

Deep conceptual understanding enables sophisticated analysis. The deeper a connection to the abstract fabric of mathematics and scientific modeling our students have, the easier they can understand the conceptual spikes in Figure 2. This in turn increases their ability to understand "hard" parts of their fields.

Neither computations, nor concepts will stand on their own. Consider an entirely fictitious person who is perfect with one and a total zero in the other. What good is it, if I can use a model, but cannot begin to create or even just understand it? Ultimately I will make a mistake, because I don't understand, and that mistake might have grave consequences. What good is it, if I can model a really complicated phenomenon if any of my applications are doomed because I cannot compute outcomes? No one will ever trust my models because I cannot derive sensible results from them.

Thankfully such lopsidedness does not exist in human beings. We usually prefer one or the other, but we have ability in both. Computationally strong people who can apply models well make excellent working engineers. They have conceptual and modeling ability, but they prefer to use it to understand and apply rather than explore and create. These are the people who build our roads and keep our lights on. Conceptually strong people who can see through a complex phenomenon and describe it mathematically make excellent scientists and engineering designers. They like to explore new approaches, create new and improved devices or sharpen our understanding of the universe through new theories and theorems. These are the people who improve fuel efficiency in cars or who find new ways to detect extrasolar planets and more details about these planets.

The topic or the individual problem usually decides the balance between computation and conceptual argumentation. There is no point in obscuring depth with computational tricks and there also is no point in looking for depth where there may be none.

Balance is the key to get both. Both need to be taught at a reasonable level, emphasizing one or the other depending on the topic. Here we can go into myriads of details. I would like to give but two examples.

Conceptual understanding also supports computations and makes some epic memorizations unnecessary. A typical Achilles heel in calculus is trigonometry. Consider the values of the trigonometric functions at special angles between 0 and 2π. It is possible to memorize the values of the trigonometric functions at all 17 angles. Unfortunately that often still is not enough if our value lies outside the interval $[0, 2\pi]$. It is easier to develop a good visual understanding of the graphs of sine and cosine and derive any value at a special angle through the appropriate shift and reflection. That is the approach taken here. The table of the unit circle with all values marked is an exercise. I recommend it to anyone more comfortable seeing information in that form.

A little extra computation can help emphasize a concept. The pinnacle of calculus is vector calculus and the integral theorems that are used in field theory, fluid dynamics, heat transfer and other areas. Surface integrals of vector fields are the core concept and it is formulated in terms of parametric surfaces. The computations with parametric surfaces can be a bit tedious, because we have to compute partial derivatives and a cross product, plug the parametrization into the field, then compute a dot product and then integrate. However, as soon as shortcut formulas for surfaces $z = f(x, y)$ are given we have to worry about which formula to apply when, rather than what we get out of the computation conceptually. I decided to present the definition in terms of parametric surfaces and do all classroom computations with that definition. If students want to use specific formulas for surfaces $z = f(x, y)$, etc., they can find them in the text and I have no problem if they use them correctly. Consequently, some of my computations are a little longer, but the conceptual foundation of the subject is reinforced in every class on vector calculus.

Both of the above ideas have influenced this text. I will not discourage memorization if it helps a student. Special formulas are usually presented as boxed exercises. If someone want to use them, why not? The idea is to get students to do mathematics correctly. There is no stipulation it has to be done a certain way, as long as it is correct.

Logic and the language of mathematics. Mathematics is challenging to read because a mathematician's use of language is different from everyday use. First, in mathematics there is no ambiguity, while in everyday language "maybe," "approximately," "usually" etc. explicitly allow for ambiguity. Second, some words, most notably "or," have a different meaning in mathematics ("or" is inclusive) than in everyday language ("or" is usually exclusive). To foster mathematical reading skills, there is a separate module on logic (cf. Module 2). Thereafter, I included exercises that require the student to state the contrapositive and the converse of theorems and to decide if the converse is true. It's a simple vehicle to make students work closely with statements that we want them to fully understand.

Integrating Precalculus. We cannot lose sight of the fact that students come to college with various levels of preparation. Many students enter Louisiana Tech University prepared to the point where precalculus may be too little, but calculus right away would be too much. When we realized that we could integrate some precalculus and still teach all necessary content in the first two years, Louisiana Tech University went to a mathematics sequence that integrates precalculus with calculus in the first three quarters. We had to expand one class by a no credit one-hour "lab" session, but the payoff was worth it. We now teach more content at a deeper level in less time, because we can rely on solid fundamentals and also because we go a bit deeper in the fundamentals. Since this text was developed in this integrated setting, it contains modules on precalculus.

Modular Design and Curriculum Integration. Independent of the text's roots, the text can be used in any setting. There is no law that says you have to start a book on page 1 and, as Figure 2 shows, mathematics is often accessed in nonlinear fashion in applications. The modular design allows any user to skip the modules that are not necessary and to cover modules in any of a large number of sensible orderings. Consider Figure 1 for the prerequisite structure of this project. For example, if your students are ready for calculus, start there and use the precalculus sections as background material.

The idea for modular design arose when designing Louisiana Tech University's integrated engineering and science curricula. Our very positive experiences are summarized for example in [4, 34]. Modular design allowed us to address the conundrum presented by Figure 2. Once the prerequisite structure of all topics was analyzed and condensed into Figure 1 it was possible to shift key topics to a better place in the curriculum. Integration and differential equations are now introduced parallel to the first physics class (mechanics), vector analysis is parallel to the second (electricity and magnetism). Statistics is introduced parallel to a materials class that emphasizes statistical analysis of real data. With these cross-connections, content settles deeper in the student's mind and is ready when needed in applications.

About the Cover

The use of an artist's rendering of a space station was inspired by Example 15.2.1 (minimum energy needed to lift a 1kg payload to the international space station) and Figure 15.23 (the international space station) as well as Examples 15.3.1 and 15.3.7 (escape velocity from earth). Within this stunning and important project, visual and symbolic aspects of the mathematical foundations of calculus and physics are "reflected". For visual aspects, the front right solar panel shows two of the four types of discontinuities of functions presented in Figure 6.59. The lines on planet Earth are the definition of the derivative as the slope of the tangent line as in Figure 9.21. The back left solar panel shows Riemann sums approaching the area under a function as in Figure 12.31. For symbolic aspects, the front left panel shows the characterization of the limit at a point via sequences in Theorem 6.5.7. The back right panel shows the power rule presented in Theorems 10.1.1 and 10.7.8 and the middle panel shows the antiderivative form of the Fundamental Theorem of Calculus presented in Theorem 14.1.1. Overall, the cover emphasizes that mathematics is at the heart of all scientific and engineering endeavors.

Acknowledgements

This work has grown over several years and many colleagues have given me valuable input on the text or made me aware of tools I could use.

Khaled Al-Agha and Galen Turner proofread early versions of some modules. Typos and suggestions were reported by many colleagues, most notably (so far) John Hunt, Jinko Kanno, Jim Marion, Dave Meng, Nathan Ponder and Natalia Zotov. Danny Eddy helped take pictures connected to chemistry and Pedro de Rosa helped take pictures connected to physics. Special thanks are due to my academic director, Gene Callens, who has been a great mentor over the years.

Numerous students have provided input through telling me about what is needed in their upper level classes as well as by reporting typos. To all these students I am grateful. In particular, I would like to thank Jason Tanner for sharing his "Li ate" approach to integration by parts (cf. p. 454) with me.

Since Fall 1998, Devery Rowland printed numerous early versions of a variety of modules. Through our conversations he also taught me a lot about about the production process and in-house printing possibilities. Ann White and her staff at the Tech book store were very accommodating in the dissemination of locally produced materials.

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Some homework problems have been extracted from an exam database made available by Kansas State University's Department of Mathematics.

Figures 1.70, 1.71 and 10.20 were provided by a major U.S. manufacturing company. Sources of other pictures not produced by the author are provided in the caption.

Felix Frazier, Amy Chapman and everyone at Fountainhead Press are a pleasure to work with and they have been highly supportive throughout the development of this project. Moreover, Fountainhead's philosophy of high quality at a reasonable price with no unnecessary add-ons exactly fits my own idea of what textbooks should be.

For all the above support I am deeply grateful. However, to no one am I more indebted than to my family. My wife, Claire, and my children, Samantha, Nicole, Haven and Mlle have seen me vanish into the attic time and again to pursue my various mathematical projects. A good bit of work was once more done in the basement of the house of my parents in law, Jean and Merle. Finally, my mother, Gerda, and my father, Siegfried, gave me the foundation on which my professional career is built. Without their love and support this work would not have been possible.

Ruston, LA, December 29, 2005

Bernd Schröder

Contents

Symbols In The Text iii
Foreword iii
To The Student iii
To The Teacher vii
About the Cover ix
Table of Contents xi

1 Functions **1**
1.1 Basic Functions and Their Graphs 2
1.2 Shifting, Stretching and Reflecting 13
1.3 Modeling with Functions 19
1.4 Quadratic Functions 24
1.5 Polynomials 31
1.6 More on Zeroes of Polynomials 40
1.7 Computer Algebra Systems (CAS) 45
1.8 More on Modeling – Optimization 49
1.9 Projects 51

2 Logic **53**
2.1 If-Then-Statements (Implications) 54
2.2 Quantified Statements 66
2.3 AND, OR and other connectives 72
2.4 Negation of a Statement 80
2.5 Projects 84

3 Trigonometric Functions – Basics **85**
3.1 Measuring Angles 86
3.2 Defining the Trigonometric Functions 92
3.3 Applications of Triangles 97
3.4 The Graphs of Sine, Cosine and Tangent 102
3.5 Amplitudes, Periods/Frequencies and Phase Shifts 114
3.6 Inverse Trigonometric Functions 121
3.7 Projects 124

4 Mixed Fundamentals **125**
4.1 Proofs by Induction 125
4.2 Summation Notation 129
4.3 Systems of Linear Equations 135
4.4 Solving Systems of Nonlinear Equations 146

5 Sequences of Numbers **149**
5.1 Sequences of Numbers 150
5.2 Limits of Sequences 155
5.3 The Exact Definition of the Limit of a Sequence 162
5.4 Subsequences – A Better Characterization of Divergence 171
5.5 Recursively Defined Sequences 176
5.6 Limits of Recursive Sequences 183
5.7 Projects 185
Review Exercises 186

6 Limits of Functions and Continuity **187**
6.1 Warm-up: Rational Functions 189
6.2 Limits of Functions at Infinity 191
6.3 Theoretical Background: The Precise Definition of the Limit at $\pm\infty$ 199
6.4 Limits of Functions at a Point 206
6.5 Theoretical Background: The Precise Definition of the Limit at a Point 213
6.6 One-Sided Limits and Vertical Asymptotes 219
6.7 Continuity 226
6.8 The Intermediate Value Theorem 234
6.9 Sketching Rational Functions Using Algebra and Limits Only . . . 241
6.10 Theoretical Background: Continuous Extensions 243
6.11 Projects 246

7 Exponential and Logarithmic Functions **247**
7.1 Exponential Functions 248
7.2 Logarithmic Functions 255
7.3 Applications – Exponential and Logarithmic Equations 263

8 Trigonometric Identities and Equations **271**
8.1 Basic Trigonometric Identities and Equations 272
8.2 Additive Identities 279
8.3 Derived Identities 284
8.4 Adding Sine and Cosine Functions 287
8.5 The Law of Sines and the Law of Cosines 289
8.6 Projects 294

9 The Derivative **295**
9.1 How to Define Tangents and Velocities? 296
9.2 The Derivative 303
9.3 Differentiable Functions 311
9.4 The Relationship Between the Function f and Its Derivative f' . . 317
9.5 Higher Derivatives 325

10 Differentiation Formulas **331**
10.1 The Power Rule and the Natural Exponential Function 332
10.2 Graphing Functions 340
10.3 Absolute Extrema and Optimization 344
10.4 The Product and Quotient Rules 349
10.5 The Chain Rule 358
10.6 The Derivatives of the Trigonometric Functions 368
10.7 Implicit Differentiation and Inverse Functions 374

11 Problem Analysis with Derivatives 383
11.1 Optimization When Parameters are Involved 384
11.2 Graphing Functions 389
11.3 Related Rates 395
11.4 L'Hôpital's Rule 400
11.5 Linear Approximations and Error Estimates 407

12 Definite Integrals 411
12.1 Eliminating Variation By Examining Small Increments 412
12.2 The Definite Integral 425
12.3 Projects 435

13 Indefinite Integrals (Antiderivatives) 437
13.1 Basic Indefinite Integrals 438
13.2 Substitution 446
13.3 Integration by Parts 453
13.4 Integration of Rational Functions 459
13.5 Deciding Which Integration Method to Use 467
13.6 Projects 472

14 The Fundamental Theorem of Calculus 473
14.1 The Fundamental Theorem of Calculus – Antiderivative Form . . 474
14.2 Computing Areas Under and Between Curves 480
14.3 The Fundamental Theorem of Calculus – Derivative Form 489

15 Applications of Integration 491
15.1 Volumes 492
15.2 Work 499
15.3 Improper Integrals I: Infinite Intervals 502
15.4 Improper Integrals II: Singularities 510
15.5 Centers of Mass of Linear Densities 513
15.6 Projects 516

16 Numerical Methods 519
16.1 Taylor Polynomials 519
16.2 Numerical Integration 528
16.3 Newton's Method 542
16.4 Projects 546

17 Statistics – Continuous Distributions (Single Variable) 547
17.1 Fundamentals on Probability Functions 548
17.2 Continuous Random Variables 554
17.3 Some Widely Used Density Functions 558
17.4 Cumulative Distribution Functions 567
17.5 Mean and Variance 571
17.6 Questions to Ask When Solving Statistics Problems 576

18 Statistics – Application to Measurements 579
18.1 Some Sample Statistics 579
18.2 Statistical Behavior of the Sample Mean 583
18.3 Confidence Intervals 591

19 Series **597**
19.1 Series of Numbers 598
19.2 Convergence Tests 603
19.3 Power Series 609
19.4 Multiplying and Dividing (Power) Series 616
19.5 Projects 620

A Tables and Formulas **621**
A.1 Some Probability Density Functions 621
A.2 The Cumulative Standard Normal Distribution $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ 622
A.3 Critical values t_α for the t-distribution 623
A.4 Geometric Formulas 624
A.5 Trigonometric Formulas 626
A.6 A Short Integral Table 627

B Technical Proofs and Technical Parts of Proofs **629**
B.1 Proving the Monotonic Sequence Theorem 630
B.2 Proof of the Intermediate Value Theorem 630
B.3 Proof of Theorem 6.8.11 630
B.4 Justifying the Definition of Powers with Arbitrary Exponents . . . 631
B.5 The Natural Exponential Function – Existence, Properties and Representations 633
B.6 Completing the Proof of Theorem 8.2.1 636
B.7 Completing the Proof of Theorem 10.1.1 637
B.8 Theoretical Background on Integration: (Piecewise) Continuous Functions are Integrable 638
B.9 Proving the Comparison Test for Improper Integrals 640
B.10 Proof of Theorem 16.1.1 641
B.11 Proof of Theorem 16.1.11 642

C References to modules that are not included **643**

D Hints and Solutions for Selected Problems **647**

Bibliography **669**

Index **671**