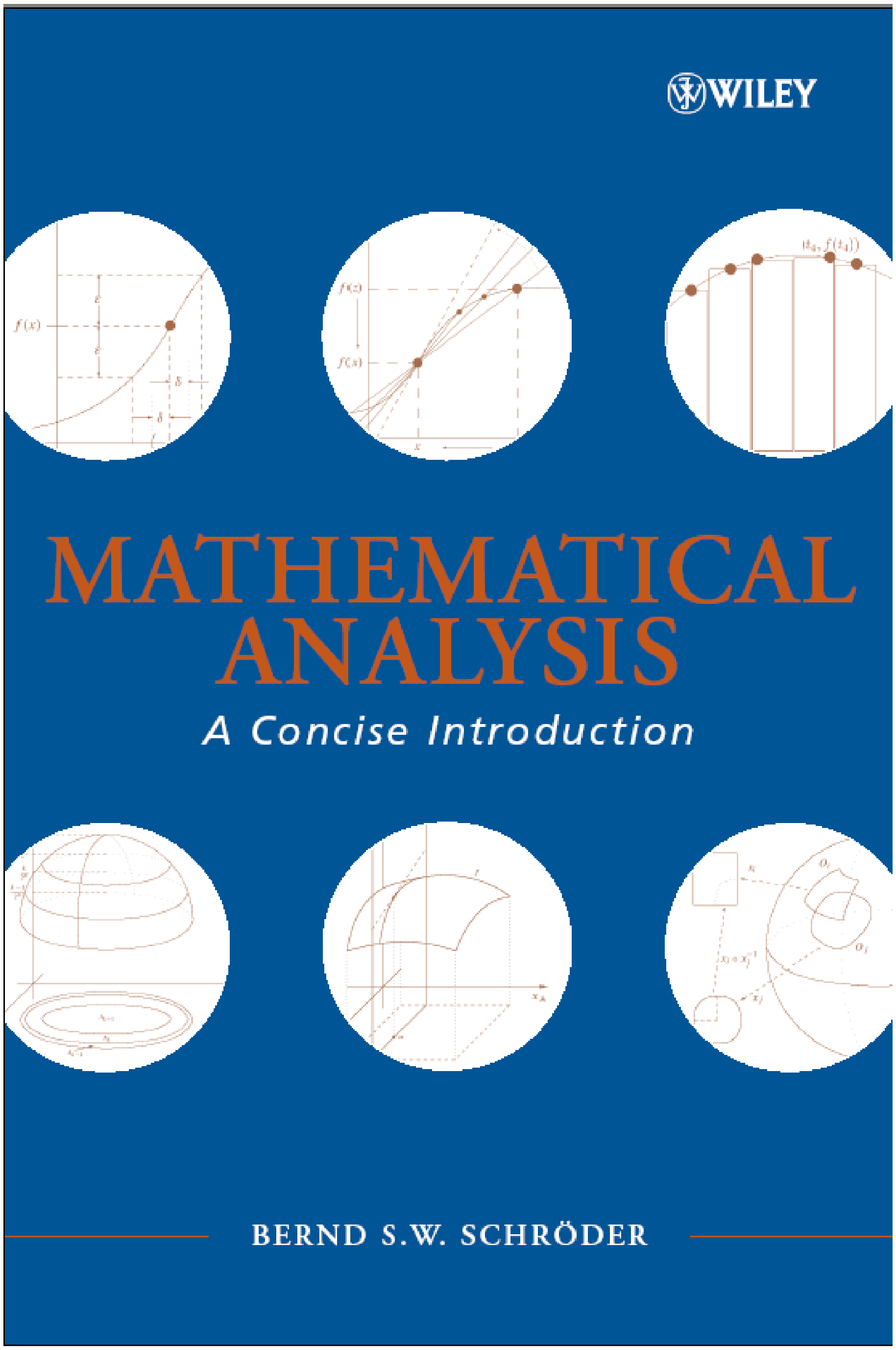

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Part II: Analysis in Abstract Spaces

Appendices

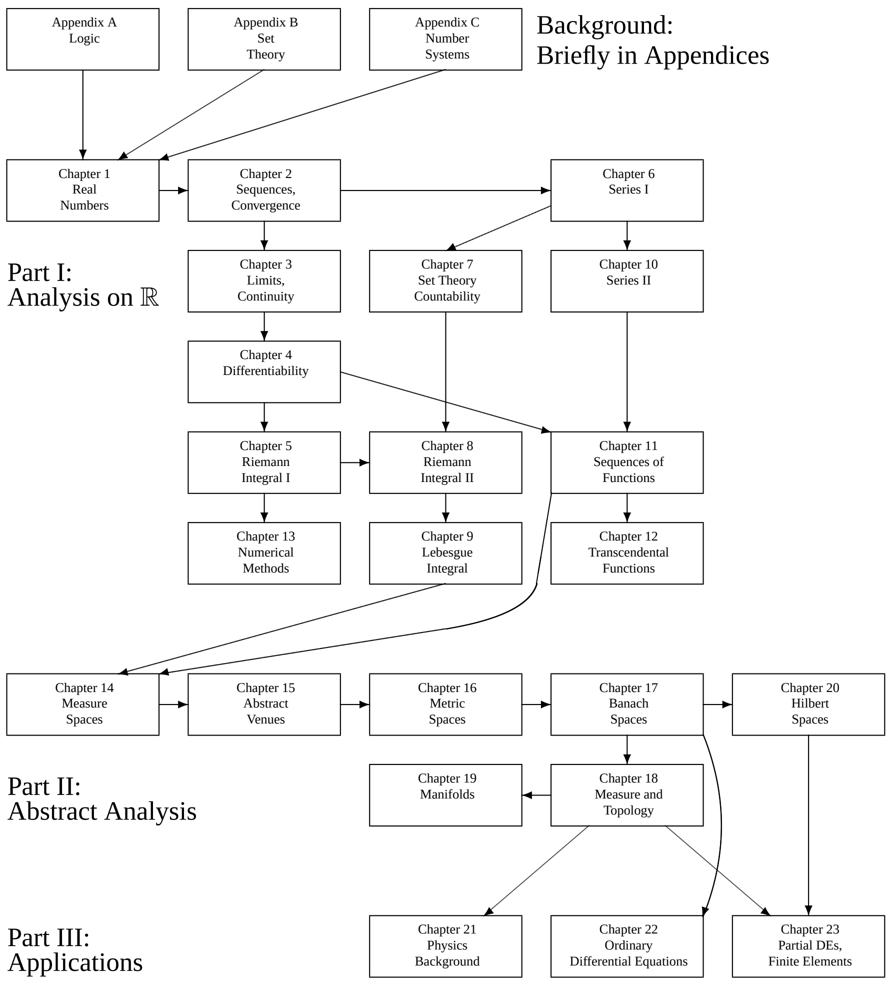

Figure 1: Content dependency chart with minimum prerequisites indicated by arrows. Some remarks, examples, and exercises in the later chapter might still depend on other earlier chapters, but this problem typically can be resolved by quoting a single result. Details about where and how the reader can "branch out" are given in boxes in the text.

Preface

This text is a self-contained introduction to the fundamentals of analysis. The only prerequisite is some experience with mathematical language and proofs. That is, it helps to be familiar with the structure of mathematical statements and with proof methods, such as direct proofs, proofs by contradiction, or induction. With some support in the right places, mostly in the early chapters, this text can also be used without prerequisites in a first proof class.

Mastering proofs in analysis is one of the key steps toward becoming a mathematician. To develop sound proof writing techniques, standard proof techniques are discussed early in the text and for a while they are pointed out explicitly. Throughout, proofs are presented with as much detail and as little hand waving as possible. This makes some proofs (for example, the density of $C[a, b]$ in $L^p[a, b]$ in Part II) notationally a bit complicated. With computers now being a regular tool in mathematics, the author considers this appropriate. When code is written for a problem, all details must be implemented, even those that are omitted in proofs. Seeing a few highly detailed proofs is reasonable preparation for such tasks. Moreover, to facilitate the transition to more abstract settings, such as measure, inner product, normed, and metric spaces, the results for single variable functions are proved using methods that translate to these abstract settings. For example, early proofs rely extensively on sequences and we also use the completeness of the real numbers rather than their order properties.

Analysis is important for applications, because it provides the abstract background that allows us to apply the full power of mathematics to scientific problems. This text shows that all abstractions are well motivated by the desire to build a strong theory that connects to specific applications. Readers who complete this text will be ready for *all* analysis-based and analysis-related subjects in mathematics, including complex analysis, differential equations, differential geometry, functional analysis, harmonic analysis, mathematical physics, measure theory, numerical analysis, partial differential equations, probability theory, and topology. Readers interested in motivation from physics are advised to browse Chapter 21, even if they have not read any of the earlier chapters.

Aside from the topics covered, readers interested in applications should note that the axiomatic approach of mathematics is similar to problem solving in other fields. In mathematics, theories are built on axioms. Similarly, in applications, models are subject to constraints. Neither the axioms, nor the constraints can be violated by the theory or model. Building a theory based on axioms fosters the reader's discipline to not make unwarranted assumptions.

Organization of the content. The text consists of three large parts. Part I, comprised of Chapters 1–13, presents the analysis of functions of one real variable, including a motivated introduction to the Lebesgue integral. Chapters 1–6 and 10–13 could be called "single variable calculus with proofs." For a smooth transition from calculus and a gradual increase in abstraction, Chapters 1–6 require very little set theory. Chapter 1 presents the properties of the real line and limits of sequences are introduced in Chapter 2. Chapters 3–5 present the fundamentals on continuity, differentiation, and (Riemann) integration in this order and Chapter 6 gives a first introduction to series.

Chapters 6–8 are motivated by the desire to further explore the Riemann integral while avoiding the excessive use of Riemann sums. This exploration is done with the Lebesgue criterion for Riemann integrability. Although this criterion requires the Lebesgue measure, the payoff is that many proofs become simpler. To quickly reach this criterion, the first presentation of series in Chapter 6 is deliberately kept short. It presents enough about series to allow the definition of Lebesgue measure. Chapter 7 presents fundamental notions of set theory. Most of these ideas are needed for Lebesgue measure, but, overall, Chapter 7 contains all the set theory needed in the remainder of the text. Chapter 8 finishes the presentation of the Riemann integral. With Lebesgue measure available, it is natural to investigate the Lebesgue integral in Chapter 9. This chapter could also be delayed to the end of Part I, but the author believes that early exposure to the crucial ideas will ease the later transition to measure spaces.

The analysis of single variable functions is finished with the rigorous introduction of the transcendental functions. The necessary background on power series is explored in Chapter 10. Chapter 11 presents some fundamentals on the convergence of sequences of functions and Chapter 12 is devoted to the transcendental functions themselves. Chapter 13 discusses general numerical methods, but transcendental functions provide a rich test bed for the methods presented.

Part I of the text can be read or presented in many orders. Figure 1 shows the prerequisite structure of the text. Prerequisites for each chapter have deliberately been kept minimal. In this fashion, the order of topics in the reader's first contact with proofs in analysis can be adapted to many readers' preferences. Most notably, the intentionally early presentation of Lebesgue integration can be postponed to the end of Part I if so desired. Throughout, the author intends to keep the reader engaged by providing motivation for all abstractions. Consequently, as Figure 1 and the table of contents indicate, some concepts and results are presented in a "just-in-time" fashion rather than in what may be considered their traditional place. If a concept is needed in an exercise before the concept is "officially" defined in the text, the concept will be defined in the exercise and in the text.

Part II, comprised of Chapters 14–20, explores how the appropriate abstractions lead to a powerful and widely applicable theoretical foundation for all branches of applied mathematics. The desire to define an integral in d-dimensional space provides a natural motivation to introduce measure spaces in Chapter 14. This chapter facilitates the transition to more abstract mathematics by frequently referring back to corresponding results for the one dimensional Lebesgue integral. The proofs of these results usually are verbatim the same as in the one-dimensional setting. Moreover, this early introduction makes L^p spaces available as examples for the rest of the text. The abstract venues of analysis are then presented in Chapter 15, which provides all examples for the rest of Part II.

The fundamentals on metric spaces and continuity are presented in Chapter 16. As with measure spaces, for several results on metric spaces the reader is referred back to the corresponding proof for single variable functions. Proofs are no longer verbatim the same and abstraction is facilitated by translating proofs from a familiar setting to the new setting while analyzing similarities and differences. In a class, the author suggests that the teacher fill in some of these proofs to demonstrate the process.

Chapter 17 presents the fundamentals on normed spaces and differentiation. Again, ideas are similar to those for functions of a single variable, but this time the abstraction goes beyond translation. With all three fundamental concepts (integration, continuity, and differentiation) available in the abstract setting, Chapter 18 shows the interrelationship between concepts presented separately before, culminating in the Multivariable Substitution Formula.

The second part is completed by a presentation of the fundamentals of analysis on manifolds, together with a physical interpretation of key concepts in Chapter 19 and by an introduction to Hilbert spaces in Chapter 20.

The remaining chapters give a brief outlook to applied subjects in which analysis is used, specifically, physics in Chapter 21, ordinary differential equations in Chapter 22, and partial differential equations and the finite element method in Chapter 23. Each of these chapters can only give a taste of its subject and I encourage the reader to go deeper into the utterly fascinating applications that lie behind part III. The mathematical preparation through this text should facilitate the transition.

It should be possible to cover the bulk of the text in a two course sequence. Although Chapters 14-16 should be read in order, depending on the available time, the pace and the choice of topics, any of Chapters 17-23 can serve as a capstone experience.

How to read this text. Mathematics in general, and analysis in particular, is not a spectator sport. It is learned by doing. To allow the reader to "do" mathematics, each section has exercises of varying degrees of difficulty. Some exercises require the adaptation of an argument in the text. These exercises are also intended to make the reader critically analyze the argument before adapting it. This is the first step toward being able to write proofs. Of course the need for very critical (and slow) reading of mathematics is nicely summed up in the old quote that "To read without a pencil is daydreaming." The reader should ask him/herself after every sentence "What does this mean? Why is this justified?" Making notes in the margin to explain the harder steps will allow the reader to answer these questions more easily in the second and third readings of a proof. So it is important to read thoroughly and slowly, to make notes and to reread as often as needed. The extensive index should help with unknown or forgotten terminology as necessary. Other exercises have hints on how to create a proof that the reader has not seen before. These exercises require the use of proof techniques in a new setting. Finally, there are also exercises without hints. Being able to create the proof with nothing but the result given is the deepest task in a mathematics course. This is not to say that exercises without hints are always the hardest and adaptations are always the easiest, but in many cases this is true. Finally, some exercises give a sequence of hints and intermediate results leading up to a famous theorem or a specific example. These exercises could also be used as mini-projects. In a class, some of them could be the basis for separate lectures that spotlight a particular theorem or example.

To get the most out of this text, the reader is encouraged to *not* look for hints and solutions in other background materials. In fact, even for proofs that are adaptations of proofs in this text, it is advantageous to try to create the proof *without* looking up the proof that is to be adapted. There is evidence that the struggle to solve a problem, which can take days for a single proof, is exactly what ultimately contributes to the development of strong skills. "Shortcuts," while pleasant, can actually diminish this development. Readers interested in quantitative evidence that shows how the struggle to acquire a skill actually can lead to deeper learning may find the article [4] quite enlightening. A better survival mechanism than shortcuts is the development of connections between newly learned content and existing knowledge. The reader will need to find these connections to his/her existing knowledge, but the structure of the text is intended to help by motivating all abstractions. Readers interested in how knowledge is activated more easily when it was learned in a known context may be interested in the article [5].

Acknowledgments. Strange as it may sound, I started writing this text in the spring of 1987, as I prepared for my oral final examination in the traditional Analysis I–III sequence in Germany. Basically, I took all topics in the sequence and arranged them in what was the most logical fashion to me at the time. Of course, these notes are, in retrospect, immature. But they did a lot to shape my abilities and they were a good source of ideas and exercises. In this respect, I am indebted to my teachers for this sequence: Professor Wegener and teaching assistant Ms. Lange for Analysis I, Professor Kutzler and teaching assistant *Herr* Böttger for Analysis II-III as well as Professor Herz in whose Differential Equations class I first saw analysis "at work." With all due respect to the other individuals, to me and many of my fellow students, the force that drove us in analysis (and beyond) was Herr Böttger. This gentleman was uncompromising in his pursuit of mathematical excellence and we feared as well as looked forward to his demanding exercise sets. He was highly respected because he was ready to spend hours with anyone who wanted to talk mathematics. Those who kept up with him were extremely well prepared for their mathematical careers. Incidentally, Dr. Ansgar Jüngel, whose notes I used for the chapter on the finite element method, took the above mentioned classes with me. The thorough preparation through these classes is the main reason why most of this text was comparatively easy to write. If this text does half as good a job as Herr Böttger did with us, it has more than achieved its purpose.

It was thrilling to test my limitations, it was humbling to find them and ultimately I was left awed once more by the beauty of mathematics. When my abilities were insufficient to proceed, I used the texts listed in the bibliography for proofs, hints or to structure the presentation. To make the reader fully concentrate on matters at hand, and to force myself to make the exposition self-contained, outside references are limited to places where results were beyond the scope of this exposition. A solid foundation will allow readers to judiciously pick their own resources for further study. Nonetheless, it is appropriate to recognize the influence of the works of a number of outstanding individuals. I used Adams [2], Renardy and Rogers [23], Yosida [33] and Zeidler [34] for Sobolev spaces, Aris [3], Cramer's `http://www.navier-stokes.net/`, and

Welty, Wicks and Wilson [31] for fluid dynamics, Chapman [6] for heat transfer, Cohn [7] for measure theory, Dieudonné [8] for differentiation in Banach spaces, Dodge [9] and Halmos [13] for set theory, Ferguson [10], Sandefur [24] and Stoer and Bulirsch [28] for numerical analysis, Halliday, Resnick and Walker [12] for elementary physics, Hewitt and Stromberg [14], Heuser [15], [16], Johnsonbaugh and Pfaffenberger [20], Lehn [22] and Stromberg [29] for general background on analysis, Heuser [17] for functional analysis, Hurd and Loeb [18] for the use of quantifiers in logic, Jüngel [21] and Šolín [25] for the finite element method, Spivak [26], [27] for manifolds, Torchinsky [30] for Fourier series, Willard [32] for topology, and the Online Encyclopaedia of Mathematics `http://eom.springer.de/` for quick checks of notation and definitions. Readers interested in further study of these subjects may wish to start with the above references.

The first draft of the manuscript was used in my analysis classes in the Winter and Spring quarters of 2007. The first class covered Chapters 1–9, the second covered Chapters 11 and 14–18 (with some strategic "fast forwards"). This setup assured that graduating students would have full exposure to the essentials of analysis on the real line and to as much abstract analysis as possible without "handwaving arguments." I am grateful to the students in these classes for keeping up with the pace, solving large numbers of homework problems, being patient with the typos we found and also for suggesting at least one order in which to present the material that I had not considered. The students' evaluations (my best ever) also reaffirmed for me that people will enjoy, or at least accept and honor, a challenge, and that an ambitious, motivated course should be the way to go. Devery Rowland once more did an excellent job printing drafts of the text for the classes.

Aside from the referees, several colleagues also commented on this text and I owe them my thanks for making it a better product. In particular, I would like to thank Natalia Zotov for some comments on an early version that significantly improved the presentation, and Ansgar Jüngel for pointing out some key references on Sobolev spaces. Although I hope that we have found all remaining errors and typos, any that remain are my responsibility and mine alone. I request readers to report errors and typos to me so I can post an errata. My contacts at Wiley, Susanne Steitz, Jacqueline Palmieri, and Melissa Yanuzzi bore with me when the stress level rose and their patience made the publishing process very smooth.

As always, this work would not have been possible without the love of my family. It is truly wonderful to be supported by individuals who accept your decision to spend large amounts of time reliving your formative years.

Finally, I was sad to learn that Herr Böttger died unexpectedly a few years after I had my last class with him. Sir, this one's for you.

Ruston, LA, August 30, 2007

Bernd Schröder