Remarks and Errata for "Ordered Sets – An Introduction"

September 8, 2009

In this list I shall collect typos, factual errors, suggestions and updates on my book on Ordered Sets as they come up. Anyone who finds typos, errors or has suggestions is welcome to make me aware of them. Typos and errors are boxed.

p.22, footnote — Joshua Hughes' MS thesis has shown (as anticipated) that ordered sets up to and including size 11 are reconstructible.

p.43 — The number of antichains in the fence with n elements is actually the $(n+2)^{\text{nd}}$ Fibonacci number. This can be seen from Proposition 2.7.5, but I did not. For an elegant presentation of this result, consider Example 2.13 in Blyth's book [Blyth].

p.46, l.32 — It should be demanded that the sum of the sizes of the components is *maximal*, not minimal.

p.46, l.36 — It should be demanded that the set S is connected.

p.51 — Another interesting and apparently open question is if the diameter of an ordered set is reconstructible.

p.60, l.18 — The definition of a maximal card may look a bit strange and the reader (just like the author) might be inclined to call $P \setminus \{m\}$ a maximal card if m was maximal. However, in reconstruction we start out with the deck and no knowledge of the set, so it is conceivable (if the reconstruction conjecture is false) to have two sets with equal decks for which a certain card is obtained by removing a maximal element in one set and a non-maximal element in the other.

Joshua Hughes pointed out to the author that the 3-element non-reconstructible sets give an example of such a situation.

p.72 — Another nice exercise for this section came up in conversation with Jerzy Wojdylo.

Proposition 3.3.7 shows that for every ordered set P there is a set S with $|S| \leq |P|$ such that P can be embedded into the power set $\mathcal{P}(S)$ ordered by inclusion.

(a) Show that if P can be embedded into $\mathcal{P}(S)$, then $|S| \geq \lceil \log_2 |P| \rceil$.

(b) Show that if S is such that an n-element chain can be embedded into $\mathcal{P}(S)$, then $|S| \geq n-1$ and that there is such an S with $|S| = n-1$.

(c) Let n be an integer. Show that for all integers k with $\lceil \log_2 |P| \rceil \leq k \leq n-1$ there is an ordered set P with $|P| = n$ such that for all sets S such that P can be embedded into $\mathcal{P}(S)$ we have that $|S| \geq k$ and there is such a set S of size k.

(d) Let n be an integer. Show that there is an ordered set P with $|P| = n$ such that for all sets S such that P can be embedded into $\mathcal{P}(S)$ we have that $|S| \geq n$.

p.76,l.24 The paper [250] announced as submitted has now appeared (cf. [Sch2]).

p.78, l.31 The definition of a good subset should read as follows. The subset Q of P is called a **good subset** iff no two elements of Q are funneled through each other and for all $p \in P$, there is $q \in Q$ such that p is funneled through q.

The retraction r onto a good subset Q then is the map that maps each P to the smallest element of Q that is above p or (if that does not exist) to the largest element of Q that is below p. (For all p, one of these two must exist.)

The proof that r is a retraction remains the same, though.

For all later work with good subsets, the main property that is used consistently is that each p is funneled through $r(p)$, so the later proofs are correct for either version.

The definition of good subsets as stated here allows for good subsets Q in which a point of P is funneled through more than one point in Q. This can happen for example if a point has a unique upper and a unique lower cover.

p.85,l.15,16 (M. Kukiela) max and min should be replaced by sup and inf, respectively.

p.87,l.11 (M. Kukiela) It should have been stated that $P_n = Q$.

p.92, l.13ff (M. Kukiela) The distance function should be called $dist(\cdot,\cdot)$, not $d(\cdot,\cdot)$.

p.98, l.16 "$p_{n_k} \not\geq C$" should be "$p^c_{n_k} \not\geq C$".

p.98, l.16 "$p^c_{n_{k_c}} \geq p$" should be "$p^c_{n_k} \geq p$"

p.98,l.26 The cardinal numbers defined in Definition 4.5.10 are also referred to as regular cardinal numbers.

pp.96-99 Lemmas 4.5.3, 4.5.9, 4.5.13 and proof of Theorem 4.5.1: Numbers of theorems in [81] which are referenced to are shifted by one. Theorem 6.7 from [81] should be Theorem 6.8, Theorem 6.8 should be Theorem 6.9, Theorem 6.9 should be Theorem 6.10 and Theorem 6.10 should be Theorem 6.11.

p.100, Figure 4.2 The c's and d's are one font size too large.

p.108, l.16 "$\Phi(p) := \{x \in P : x \not\sim p\}$" should read "$\Phi(p) := \{x \in R : x \not\sim p\}$"

p.132, l.8 Even though the bottom element $\mathbf{0}$ is the supremum of the empty set of atoms, the definition would better read as one of the following.

- A lattice L is called **atomic** iff every element of $L \setminus \{\mathbf{0}\}$ is a supremum of atoms.

  Or:

- A lattice L is called **atomic** iff every element of L is a supremum of a set of atoms.

p.133, l.1 Not an error, but possibly interesting follow-up questions.

(J. Hughes) If P and Q are ordered sets and there are order-preserving *surjective* (rather than injective) functions $f : P \to Q$ and $g : Q \to P$, must P and Q be isomorphic?

This answer is "no" with an example being two ordered sets P and Q obtained from the one-way infinite fence $F = \{ f_0 < f_1 > f_2 < f_3 \cdots \}$ as follows. To obtain P we attach to each f_{2k} $2k$ more upper covers $u_1^{2k}, \cdots, u_{2k}^{2k}$ such that f_{2k} is their unique lower cover. To obtain Q we attach to each f_{2k} $2k-1$ more upper covers $u_1^{2k}, \cdots, u_{2k-1}^{2k}$ such that f_{2k} is their unique lower cover. Clearly P and Q are not isomorphic.

The map that is the identity on F and maps the additional upper covers surjectively onto each other is a surjective order preserving map from P to Q. The map that shifts F one to the left ($f_k \mapsto f_{k-2}$), mapping f_0, f_1 and the additional upper cover of f_2 to f_0 and which maps the additional upper covers surjectively onto each other is a surjective order preserving map from Q to P.

It can be concluded however, that if surjective maps as above exist, then P and Q must have equal width.

(Extension.) If P and Q are ordered sets and there are order-preserving *bijective* (rather than injective or surjective alone) functions $f : P \to Q$ and $g : Q \to P$, must P and Q be isomorphic?

For finite sets, this is trivial, but what about infinite sets?

p.142, l.13 (M. Kukiela) "G = (V, P)" should be "G = (V, E)"

p.149,Fig.6.2 The Fixed Point Property for an ordered set is not equivalent to the Fixed Clique Property for the comparability graph by Example 6.3.6.

The Fixed Clique Property for a graph is equivalent to the Fixed Simplex Property for the induced simplicial complex. This is because for complexes induced by graphs, simplicial maps are exactly the graph homomorphisms.

By Example 5 the topological fixed point property for the topological realization of a simplicial complex is not equivalent to the fixed simplex property. Is there a simpler way to come up with examples such as Example 5 than what is shown here?

p.158, l.19 The statement "It has been proved in [128] that if two ordered sets have the same comparability graph and the same decks, then they must be isomorphic. Thus graph reconstruction implies order reconstruction." is **completely wrong**. It should actually read "It has been proved in [128] that if two ordered sets P, P' with more than 3 elements have the same comparability graph and $P \setminus \{x\}$ is isomorphic to $P' \setminus \{x\}$ for all x, then they must be isomorphic. Thus graph reconstruction *almost* implies order reconstruction." The statement as it is in the book would have shown that graph reconstruction implies order reconstruction. It just has not been proved (yet?).

Embarrassingly enough, I specifically point out this detail in my MR review of the paper in question. Subsequently that detail seems to have slipped out of my conscience. While the lapse is unexcusable, I must say that I find it highly likely that graph reconstruction implies order reconstruction. The results in Section 9.4, specifically Proposition 9.4.4, Lemma 9.4.5 and Theorem 9.4.6, are a strong indication that decomposable ordered sets should be reconstructible (the results almost show it). This would reduce a proof of graph reconstruction implying order reconstruction to proving it for indecomposable sets, which amounts to using the deck to distinguish between the set and its dual. (This is easier said than done, but it does not sound impossible.)

p.160,l.26 In [LLP] ordered sets are presented which have the fixed point property and for which the size of the iterated clique graphs goes to infinity. Thus the fixed point property is not equivalent to the "collapsing of the iterated clique graphs" (called K-null in [LLP]). There are three possibilities for the sequence of iterated clique graphs. Either the sequence ends with the trivial graph ("K-null"), or the sequence ends in a cycle of nontrivial graphs ("K-bounded") or the size of the iterated clique graphs goes to infinity ("K-divergent"). We know now that K-nullity of the comparability graph implies the fixed point property and that there are ordered sets with K-divergent comparability graphs and the fixed point property. For the relationship between the fixed point property and K-boundedness we can say that crowns do not have the fixed point property and their comparability graphs are K-bounded. The question remains if there is an ordered set with the fixed point property and a K-bounded comparability graph.

p.180, l.1 The statement in Exercise 18 is wrong as it stands. In the ordered set $0 \leq a, b \leq 1$, with a, b not comparable, removal of a or b both decrease the dimension by 1. (Observation of Mr. Bhavale Ashok Nivrutti of the University of Pune.)

I'm not sure what a satisfactory alternative statement might be.

p.187 A simpler, inductive proof of Theorem 8.1.5 and of the finite version of the Scott-Suppes theorem (p.197, Exercise 11a) are given in [BalBog].

p.193, l.33 (M. Kukiela) Instead of “The standard example shows ...”, the sentence should start with “Example 8.1.4 shows ...”

p.206, l.3 (M. Kukiela) It should be $F(t) := I[f[P_t]]$ rather than $F(t) := I[P_t]$.

p.216, l.5 “card crucial” should read “crucial card”

p.217,l.1 In his MS thesis, J. Hughes provides a computer check that ordered sets with up to 11 elements are reconstructible. The program for size 11 is still running (see footnote on p.22), so barring a sensation, the size restriction $|P| \geq 12$ in Lemma 9.4.5 and Theorem 9.4.6 can be replaced with the more customary $|P| \geq 4$.

p.219 Another nice exercise for this section is to investigate ranked sums. The ranked sum $R\{P_t | t \in T\}$ of a set of pairwise disjoint finite ordered sets P_t over the finite index set T is the union $\bigcup_{t \in T} P_t$ ordered as follows. Let $a \in P_s$, $b \in P_t$. We say $a \leq b$ iff $s = t$ and $a \leq_t b$ or $s < t$ and $\text{rank}_{P_s}(a) \leq \text{rank}_{P_t}(b)$.

Possible exercises.

Show that the order of the ranked sum is contained in the order of the corresponding lexicographic sum.

Show that the ranked sum of two ordered sets, one of which has the fixed point property, over a 2-chain need not have the fixed point property. (This means there is no analogue of Lemma 9.19 for ranked sums.)

The following questions should not be hard, but I have not thought of them very long. Their answers will also make nice exercises.

Are there two ordered sets with the fixed point property such that their ranked sum over a 2-chain does not have the fixed point property?

Are there reconstruction results for ranked sums analogous to those for lexicographic sums?

p.220, l.14 (M. Kukiela) “relational fixed point property” should read “connected relational fixed point property”

p.225, l.9 There is at least one more another covering graph invariant: It’s the isometric order embeddability into a Boolean lattice, see [Wi].

p.227, l.17 (M. Kukiela) “from P to Q” should read “from Q to P”

p.233, l.30 (M. Kukiela) The capital Π symbols should be lowercase π.

p.239, l.3 (M. Kukiela) The capital Π symbol should be lowercase π.

p.258, l.15 The hiint should read “For each p there are $k, m \in \mathbb{N}$ with $f^k(p) \sim f^{k+m}(p)$; also use Exercise 24.”

p.259ff It is possible to show that the fixed clique and fixed simplex properties in the finite case are preserved by the strong product of graphs (cf. Definition 1), resp. simplicial complexes (cf. Remark 3), thus improving Theorem 4.8 in [15]. The fixed clique property is not preserved by the cartesian product of graphs as the cartesian product of two paths of length 1 is a four-cycle, which does not have the fixed clique property.

Definition 1 *Let $G = (V, E)$ and $G' = (V', E')$ be graphs. We define $G \times G'$ to be the graph with vertex set $V \times V'$ and edge set*

$$E_{G\times G'} := \{\{(a,b),(x,y)\} : a, x \in V,\ b, y \in V',\ a \sim x,\ b \sim y\}.$$

Theorem 2 *(Compare with Théorème 4.8 in [15]) Let $G = (V, E)$ and $G' = (V', E')$ be finite graphs with the fixed clique property. Then $G \times G'$ has the fixed clique property.*

Proof. Let $(A, B) \in TCL(G) \times TCL(G')$. Then we define

$$\Phi((A, B)) := A \times B \in TCL(G \times G').$$

Clearly Φ is an isomorphism from $TCL(G) \times TCL(G')$ onto a subset of $TCL(G \times G')$.

Now let $\pi : V \times V' \to V$ and $\pi' : V \times V' \to V'$ be the natural projections. Then the map

$$r : TCL(G \times G') \to TCL(G \times G'); \qquad X \mapsto \pi[X] \times \pi'[X]$$

is a retraction on $TCL(G \times G')$. First of all r is well-defined, since if X is a clique in $G \times G'$, then $\pi[X]$ is a clique in G and $\pi'[X]$ is a clique in G'. The facts that r is order-preserving and idempotent are now trivial. Moreover clearly $X \subseteq r[X]$ for all $X \in TCL(G \times G')$ and $r[TCL(G \times G')] = \Phi[TCL(G) \times TCL(G')]$.

Since the graphs G, G' have the fixed clique property, the ordered sets $TCL(G)$, $TCL(G')$ have the fixed point property. Thus since all sets involved are finite by Theorem 1.1 in [228] the ordered set $TCL(G) \times TCL(G')$ has the fixed point property. Thus $r[TCL(G \times G')]$ has the fixed point property. Since r is a comparative retraction, this means that $TCL(G \times G')$ has the fixed point property and hence $G \times G'$ has the fixed clique property. ■

Remark 3 In a fashion similar to Theorem 2 we can prove that the fixed simplex property is productive, where the product of two simplicial complexes $(V_1, \mathcal{S}_1)$ and $(V_2, \mathcal{S}_2)$ is the simplicial complex $(V_1 \times V_2, \mathcal{S})$, with $\mathcal{S}$ being the set of all subsets of sets $S_1 \times S_2$, where $S_i \in \mathcal{S}_i$.

This observation is the key to Example 5. ■

Lemma 4 *Let H and K be two simplicial complexes. Then $|H| \times |K|$ is a topological retract of $|H \times K|$.*

Proof. The topological realization $|H \times K|$ is the set of all maps $\gamma : V_H \times V_K \to [0, 1]$ such that

(a) $\{(v, w) \in V_H \times V_K : \gamma(v, w) \neq 0\} \in \mathcal{S}_{H \times K}$,

(b) $\displaystyle\sum_{(v,w)\in V_H \times V_K} \gamma(v, w) = 1$,

with the componentwise metric. The product $|H| \times |K|$ is the set of all pairs (α, β), where α is an element of the topological realization of H and β is an element of the topological realization of K, with the appropriate "component-componentwise" metric. The map Φ defined by $\Phi[\alpha, \beta](v, w) = \alpha(v)\beta(w)$ maps every point in $|H|\times|K|$ to a point in $|H\times K|$. Φ is injective and continuous both ways, which means Φ is a topological isomorphism between $|H| \times |K|$and $\Phi[|H| \times |K|]$.

Define $r : |H \times K| \to \Phi[|H| \times |K|]$ as follows. For $\gamma \in |H \times K|$ set $\alpha_\gamma(v) := \displaystyle\sum_{b\in V_K} \gamma(v, b)$ for $v \in V_H$ and set $\beta_\gamma(w) := \displaystyle\sum_{a\in V_H} \gamma(a, w)$ for $w \in V_K$. Now set $r(\gamma) := \Phi(\alpha_\gamma, \beta_\gamma)$. Then r is a topological retraction of $|H \times K|$ onto $\Phi[|H| \times |K|]$. ■

Example 5 *There is a simplicial complex K that has the fixed simplex property and whose topological realization does not have the topological fixed point property.*

We shall prove the claim by contradiction. Suppose for every simplicial complex K the fixed simplex property for K would imply the topological fixed point property for the topological realization $|K|$. Let A and B be compact submanifolds of $\mathbb{R}^n$ with the fixed point property. Let H be a triangulation of A and let K be a triangulation of B. Then H and K and hence by Remark 3 $H \times K$ have the fixed simplex property. This would mean (by our assumption) that $|H \times K|$ has the topological fixed point property. By Lemma 4 this implies that $|H| \times |K|$ and hence $A \times B$ has the topological fixed point property. This would mean that the product of any two compact manifolds with the topological fixed point property again has the topological fixed point property, a contradiction to the example in Section 5 of [33].

In particular, the above means that the product discussed in Section 5 of [33] yields an example as we desire. ■

p.260, l.33 (M. Kukiela) Actually both factors should have the fixed point property and not the strong fixed point property.

p.262, l.6 Problem 18 was inspired through conversations with Jonathan Farley. I must have subsequently forgotten the inspiration. When I wrote the chapter the problem seemed natural to include, though.

p.308, l.9 (M. Kukiela) The function γ must be so that *adjacent vertices have distinct images*, of course.

p.310, footnote (M. Kukiela) "has be be" should read "has to be"

p.359ff Some references that were near publication came out after completion of the book. The complete references can be found here.

- Reference [213] in the book is reference [JXRsurvey] here.
- Reference [249] in the book is reference [Sch1] here.
- Reference [250] in the book is reference [Sch2] here.

References

[BalBog] B. Balof and K. Bogart (2003), Simple Inductive Proofs of the Fishburn and Mirkin Theorem and the ScottSuppes Theorem, *Order* 20, 49–51

[Blyth] T. Blyth (2005), *Lattices and ordered algebraic structures*, Springer, London

[LLP] F.Larrión, V. Neumann-Lara, M. Pizaña (2004), Clique divergent clock-work graphs and partial orders, *Discrete Applied Mathematics* 141, 195–207

[JXRsurvey] J.-X. Rampon (2005), What is reconstruction for ordered sets?, *Discrete Math.* 291, 191–233

[RoWid3] M. Roddy (2002), Fixed points and products: width 3, *Order* 19, 319–326

[RoUlt] M. Roddy (2002), On an example of Rutkowski and Schröder, *Order* 19, 365–366

[Sch1] B. Schröder (2002), Examples on Ordered Set Reconstruction, *Order* 19, 283–294

[Sch2] B. Schröder (2003), On Ordered Sets with Isomorphic Marked Maximal Cards, *Order* 20, 299–327

[Wi] M. Wild (1992), Cover-preserving order embeddings into Boolean lattices, *Order* 9, 209–232