Remarks and Errata for "Mathematical Analysis – A Concise Introduction"

April 28, 2010

This document collects formal and typographical errors, as well as comments, exercises, explanations and figures that I discovered, or which were mentioned to me, after publication of the text. For references boxed in green, corrected pages were submitted after the first printing. So chances are that they are corrected if you purchased the text after summer 2008.

References to remaining errors are boxed in red. Comments etc. are not specially marked. These errors as well as additional ideas could only be fixed/incorporated in a new edition.

Reports of typos/errors as well as suggestions and comments are welcome. I will keep an up-to-date errata posted, and I apologize for each and every one of my mistakes.

In the listing below, the base is the page number, a superscript indicates lines counted from the top, and a subscript indicates lines counted from the bottom.

8^{17} Exercise 1-11a should read "Let $x, y \in \mathbb{R}$. Prove that if $x < 0$ and $y \geq 0$, then $|xy| = |x||y|$."

$10_{8,9,10,13,16}$ It should be stated that the sets are nonempty.

24_5 This exercise could be extended to proving the inequality between arithmetic and geometric means in general as follows.

Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n \geq 0$. Prove that $\sqrt[n]{a_1 \cdots a_n} \leq \dfrac{a_1 + \cdots + a_n}{n}$.

Hint. First prove by induction that the result holds for $n = 2^m$. Then let $2^m \geq n$, $b_1 = a_1, \ldots, b_n = a_n$ and $b_{n+1} = \cdots b_{2^m} = \dfrac{a_1 + \cdots + a_n}{n}$ and apply the result.

35^{18} 2-8 part c is a bit more involved than meets the eye, because formally we do not have the limit law for the square root yet. I intended for that law to be used (it's proved in 2-13).

35^{18} The sequence $\left\{n - \sqrt{n^2 + 2n}\right\}_{n=1}^{\infty}$ would be another nice part for exercise 2-8.

35_1 Another part for exercise 2-16 could be to ask for an example of a bounded sequence $\{a_n\}_{n=1}^{\infty}$ so that $\left\{\dfrac{1}{n}\sum_{k=1}^{n} a_k\right\}_{n=1}^{\infty}$ diverges.

37^6 It should be noted that we can assume without loss of generality that $F \subseteq I$. Problem was resolved by focusing on $F \cap I$ in the proof.

37^8 It could be pointed out explicitly that $I \setminus F \neq \emptyset$ in this case, because otherwise $F = I$, which is not possible.

41^{20} The last inequality should be $|a_n - L| < \varepsilon$, not $|a_N - L| < \varepsilon$.

55^{3} In connection with the theorem, the following would be a nice exercise. It sounds surprising, but once the omitted part of the hypothesis of Theorem 3.14 is identified, the fog lifts.

Let $I, J \subseteq \mathbb{R}$ be open intervals, let $x \in I$, let $g : I \setminus \{x\} \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions with $g\big[I \setminus \{x\}\big] \subseteq J$, and let $\lim\limits_{z \to x} g(z) = L \in J$. Prove that even if $\lim\limits_{y \to L} f(y)$ exists it may <u>not</u> be the case that $f \circ g$ converges at x.

55_{5} The exercise would read better as "Prove that $\lim\limits_{z\to x} f(z)$ exists iff $\lim\limits_{h \to 0} f(x+h)$ exists and in this case $\lim\limits_{z\to x} f(z) = \lim\limits_{h\to 0} f(x+h)$."

68^{16} It should have been added that for $x < 0$ and q odd we define $x^{\frac{1}{q}}$ to be $-|x|^{\frac{1}{q}}$.

79_{3} A nice lemma or hint would be to first prove that $\frac{d^n}{dx^n} x f(x) = x \frac{d^n}{dx^n} f(x) + n \frac{d^{n-1}}{dx^{n-1}} f(x)$.

$86^{Fig.14}$ The bottoms of the rectangles do not quite line up with the x-axis.

97_{8} Setting $\delta := \min\left\{ \frac{\varepsilon}{4n(M+1)}, \frac{\|P\|}{3} \right\}$ avoids any possible confusion between Δx's for P and Δx's for Q.

97_{5} $Q : S \cup P$ should read $Q := S \cup P$

107^{11} The first sentence of the 2^k test should read "Let $\{a_j\}_{j=1}^{\infty}$ be a nonincreasing sequence with nonnegative terms."

111 Figure A could help explain the proof of the "$\Rightarrow$" part of Theorem 6.18.

112 Figure B could help explain the proof of the "$\Leftarrow$" part of Theorem 6.18.

114 Figure C could help explain another proof of Proposition 6.22. For a clean proof, the final estimate would need to be fleshed out a little bit, similar to the proof of how absolute convergence implies unconditional convergence.

120 The general distributive laws would be a nice exercise here.

Assume that the product $\prod\limits_{i \in I} J_i$, defined as the set of all functions $f : I \to \bigcup\limits_{i\in I} J_i$ with $f(i) \in J_i$ for all $i \in I$ is not empty. (This is true as long as we assume the Axiom of Choice holds.) Let $\{J_i\}_{i \in I}$ be a family of index sets and let $\{C_{ij}\}_{i\in I, j \in J_i}$ be a family of sets.

(a) Prove that $\bigcap\limits_{i\in I} \bigcup\limits_{j \in J_i} C_{ij} = \bigcup\limits_{f \in \prod_{i\in I} J_i} \bigcap\limits_{i \in I} C_{if(i)}$

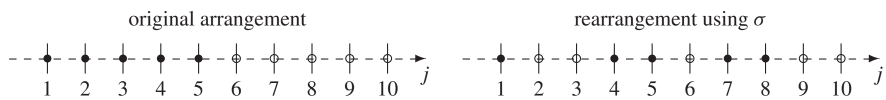

Figure A: To prove that absolute convergence implies unconditional convergence, note that if N is chosen large enough (6 in the figure), then the sum $\sum_{j=N}^{\infty} |a_j|$ is small. When the series is rearranged, the first $N-1$ terms in the unrearranged series (represented by the solid dots in the figure) must be in some set $\sigma\big[\{1, \ldots, I\}\big]$ ($I = 8$ in the figure). But then for $m \geq I$ the sum of the scrambled terms $\sum_{j=1}^{m} a_{\sigma(j)}$ must include the sum of the first $N-1$ terms a_j. Hence this sum must be close to $\sum_{j=1}^{\infty} a_j$.

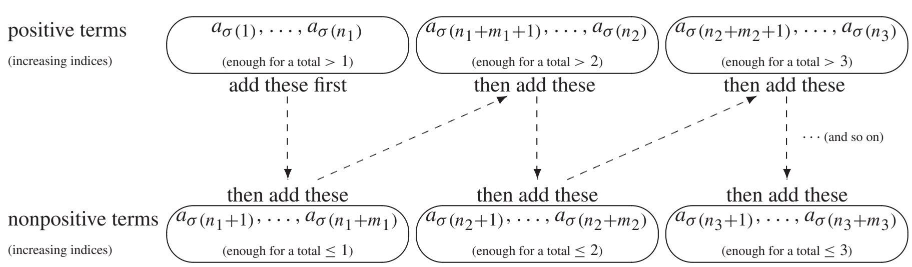

Figure B: Visualization of the proof that a convergent, but not absolutely convergent, series does not converge unconditionally.

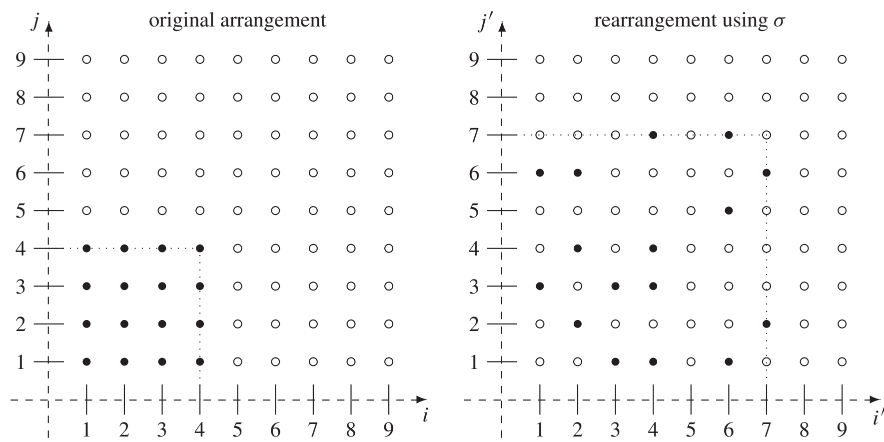

Figure C: Illustration of an alternative proof of Proposition 6.22, which is similar to the proof that absolute convergence implies unconditional convergence. Let $\varepsilon > 0$. On the left, the original arrangement of the a_{ij} visualized as dots on an $\mathbb{N} \times \mathbb{N}$ grid. In this arrangement, all sums beyond a certain cutoff m in i (here 4) add up to less than $\frac{\varepsilon}{2}$. Moreover, in the first m "vertical" sums we can find a cutoff n (here 4, too) so that for all $i = 1, \ldots, m$ we have $\sum_{j=1}^{\infty} a_{ij} < \frac{\varepsilon}{2m}$. But then there are cutoffs M for i' and N for j' for the scrambled double series (here 7 in either direction) so that every (i, j) with $i \leq m$ and $j \leq n$ is equal to some $\sigma(i', j')$ with $i' \leq M$ and $j' \leq N$ (represented with solid dots). Noting that the sums $\sum_{j=1}^{\infty} a_{\sigma(i,j)}$ are all bounded by $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}$ and must hence converge, we infer that $\left|\sum_{i'=1}^{M}\sum_{j'=1}^{\infty} a_{\sigma(i',j')} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\right| \leq \sum_{i=1}^{m}\sum_{j=n}^{\infty} a_{ij} + \sum_{i=m}^{\infty}\sum_{j=1}^{\infty} a_{ij} < \sum_{i=1}^{m} \frac{\varepsilon}{2m} + \frac{\varepsilon}{2} = \varepsilon.$

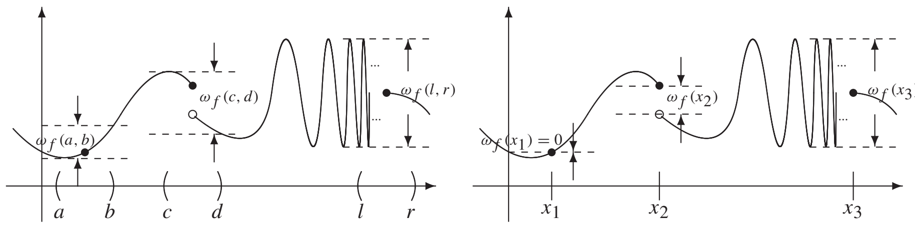

Figure D: The oscillation over an open interval is the supremum of the differences between two values in the image of the interval. The oscillation at a point is the infimum over all the oscillations over open intervals that contain the point. The point x_1 illustrates that, although the oscillations over intervals may be all positive, if f is continuous at a point, the oscillation at that point is zero. The point x_2 illustrates how the oscillation at a jump discontinuity can be the height of the jump, if the value of f at the point is between the left-sided and the right-sided limit of f at the point. The point x_3 illustrates that the oscillation at a discontinuity by oscillation is twice the amplitude of the oscillation near the point, assuming that the value of f at the point is between the largest and smallest limits of the oscillation.

(b) Prove that $\bigcup_{i\in I}\bigcap_{j\in J_i} C_{ij} = \bigcap_{f\in\prod_{i\in I} J_i}\bigcup_{i\in I} C_{if(i)}$

122_{13} "Then $f^{-1}[S]\cap\{1,\ldots,b-1\}=\{1,\ldots,n_k\}$ (with $n_0=0$ in case $k=0$) and ..." should read "Then by definition of n_1 we have $k\geq 1$, so $f^{-1}[S]\cap\{1,\ldots,b-1\}=\{n_1,\ldots,n_k\}$ and ..."

123^{11} $\bigcup_{n=1}^{\infty} C_n$ should read $\bigcup_{n=1}^{\alpha} C_n$

131 Another nice exercise would be the following. Give an example of an open interval O and a family of closed intervals $\{C_i\}_{i\in I}$ so that $O\subseteq\bigcup_{i\in I} C_i$ and there is no finite subfamily of $\{C_i\}_{i\in I}$ whose union contains O.

132 Figure D could help explain the idea behind the definition of the oscillation.

136^{10} The "The" should be a "Then".

158_{16} The range of the function g should be $[-\infty,\infty]$, not $\mathbb{R}$.

174_1 Exercise 10-8(c) would be better if it asked to prove that $\limsup_{j\to\infty}\sqrt[j]{|a_j|}>1$.

178_7 Exercise 10-16 would be easier if we considered the power series $\sum_{k=1}^{\infty} kx^k$ instead of $\sum_{k=1}^{\infty} \frac{x^k}{k}$. To be honest, that was what I meant to do, but I messed up the reciprocal. Then again, a hint that cannot be used verbatim may lead to deeper thinking about the problem. Teachers can also detect an overly strong dependence on hints or memorized patterns with problems that are "slightly off" like this one.

189_6 The proof of Theorem 12.2 would read cleaner if it said the following.

... we obtain for all k that

$$\frac{1}{\sqrt[k]{\frac{1}{k!}}} = \sqrt[k]{k!} \geq \sqrt[k]{\left\lfloor \frac{k}{2} \right\rfloor^{\left\lfloor \frac{k}{2} \right\rfloor}} \geq \left\lfloor \frac{k}{2} \right\rfloor^{\left\lfloor \frac{k}{2} \right\rfloor \frac{1}{k}} \geq \left\lfloor \frac{k}{2} \right\rfloor^{\frac{1}{3}} \to \infty \quad (k \to \infty),$$

which proves that $\infty = \liminf_{k\to\infty} \frac{1}{\sqrt[k]{\frac{1}{k!}}} = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k!}}}$ is the radius of convergence.

193^7 The following would be nice additional parts for Exercise 12-10.

(g) For fixed $a, b > 0$ and $\alpha \in \mathbb{R}$ define $\Gamma_{a,b}(\alpha) := \int_a^b x^{\alpha-1} e^{-x}\, dx$.

i. Prove that $\Gamma_{a,b}$ is continuous on $\mathbb{R}$.

ii. Prove that $\Gamma_{a,b}$ is differentiable on $\mathbb{R}$ with $\Gamma'_{a,b}(\alpha) = \int_a^b \ln|x| x^{\alpha-1} e^{-x}\, dx$.

Hint. Use Theorem 4.5 with $x = \alpha$ and $z = \alpha + \delta$. Factor out $x^{\alpha-1}e^{-x}$ in the integrand. Prove that remaining sum takes its maximum at either a or b. Then use Theorem 4.5 applied to $g(u) = b^u$ or $g(u) = a^u$ at $u = 0$.

iii. Prove that the function $\Gamma_{a,b}$ is n times differentiable on the real line with n^{th} derivative $\Gamma^{(n)}_{a,b}(\alpha) = \int_a^b \left(\ln|x|\right)^n x^{\alpha-1} e^{-x}\, dx$.

(h) Prove that the gamma function is continuous.

Hint. $\Gamma = \lim_{n\to\infty} \Gamma_{\frac{1}{n},n}$ and the limit is uniform on intervals $[c, d]$ with $d > c > 0$.

(i) Prove that the gamma function is differentiable on the interval $(0, \infty)$ with derivative with $\Gamma'(\alpha) = \int_0^\infty \ln|x| x^{\alpha-1} e^{-x}\, dx$.

Hint. Theorem 11.11.

(j) Prove that the gamma function is n times differentiable on the interval $(0, \infty)$ with n^{th} derivative $\Gamma^{(n)}(\alpha) = \int_0^\infty \left(\ln|x|\right)^n x^{\alpha-1} e^{-x}\, dx$.

197^{15} The parenthetical remark would read better as "Another integral *needed* for integration with ..."

200^{11} The hypothesis $g' \neq 0$ should be added.

200_4 The argument given here only works if $a \in \mathbb{R}$ and if we set $f(a) = g(a) = 0$, which we would be allowed to do. The argument for $a = \infty$ is similar to (actually simpler than) the argument that follows for $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$.

201_8 The sentence that starts with "We claim ... " would read better if it started with "We will prove ..." or if it ended in a colon. The result is not claimed to be true at this stage, but rather will be proved in the following.

207_{15} Exercise 13-5(d) should read $f(x) = xe^{ax}$, $f^{(n)}(x) = a^n xe^{ax} + na^{n-1}e^{ax}$.

$211^{7,8}$ The inequalities should be non-strict inequalities.

212^{16} It would be better to have $\alpha = 1$ and $h \leq \frac{1}{2}$ for the argument, because this can be achieved by choosing x_0 sufficiently close to a zero of f.

227_6 The set should be M, not X.

229^{13} The first sentence of the proof should read "By Proposition 14.12, *measurable* subsets of null sets are null sets."

230^{13} In Exercise 14-8, at least one of the sets must be demanded to have finite measure.

231^5 Definition 14.16 should say that μ is countably *sub*additive.

239^1 The function should be f throughout the definition, so the definition of the integral over A should read $\int_A f \, d\mu := \int_M f \mathbf{1}_A \, d\mu$.

242^7 The last condition in Exercise 14-32 should be $\left| f_n(x) \right| \leq B$ instead of $f_n(x) \leq B$.

$256^{15,17}$ Both lines should end with a period instead of a comma.

256_1 The "$x \in X$" should read "$x \in D$".

271_{13} Young's inequality is $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.

271_{10} We must set $v = y^q$.

278_{13} The d should be a d^s.

281^{10} Another nice exercise would be to show that the L^p-norms are nondecreasing in p when $\mu(M) = 1$.

284^9 The domain of the function should be M, not X.

336_{13} Reversing the roles of z and x is a bit harder than it looks. To make it work, we need to choose z so that $d(x,z) < \frac{r_x}{2}$, not just $d(x,z) < r_x$ as claimed on the next line. For these z and all $r \in \left(0, \frac{r_x}{2}\right)$ we have $\overline{B_r(z)} \subseteq \overline{B_{r_x-\delta}(x)}$ for some $\delta > 0$ and so $\overline{B_r(z)}$ is compact. *Now* the argument can be duplicated with x and z reversed.

336_1 The last paragraph of the proof should start with "Now let $x_0 \in X$ be arbitrary and let $C_1 := \{x_0\}$." The letter x is later used in a generic capacity, so we must formally distinguish this fixed element from the later use of x.

$353^{Fig.42}$ In the 3d view the dotted lines in the dashed rectangle spill over the rim of the rectangle on the right and on the bottom.

381^{7-12} The argument would be easier as follows.

Let $p := \lim_{n\to\infty} f^n(x)$. Because f is (Lipschitz) continuous we obtain

$$f(p) = f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = p.$$

381_5 We need to demand that $D_Y f(x,y)$ is a linear homeomorphism, not just invertible. The Open Map Theorem in Functional Analysis shows that the two are actually equivalent, but this result is not in the book.

382_1 (N. Naksinehaboon) The definition of $a(z)$ contains an extra $[z-x]$ that should not be there. The quantity $a(z)$ should actually be

$$a := 2\left\| -\big(D_Y f(x, g(x))\big)^{-1} D_X f(x, g(x))\right\| + 1.$$

Appropriate modifications need to be made at the top of p.383.

383_{12} The proof that g is the inverse of f is not totally trivial. Up to this point we have shown that $f(g(y)) = y$ for all y in a neighborhood of $f(x_0)$. In particular, f is surjective onto a neighborhood of $f(x_0)$ and g is injective. Now we would need to do the same argument for g (which can be done because by Chain Rule the derivative of g at $f(x_0)$ is a linear homeomorphism) to obtain $g(h(x)) = x$ for all x in a neighborhood of x_0 with some function h and $f(x) = f\big(g(h(x)\big) = f\circ g(h(x)) = h(x)$ shows $f = h$ and that f and g are inverses.

384^8 It should be pointed out that the $(v_1, \ldots, v_d)$ need not be a standard base representation of a vector $v \in \mathbb{R}^d$. Indeed the proof shows that the v_i may be some permutation of the components of a standard base representation.

396^{18-22} The function whose partial derivative is computed is f_φ, not f. The only f that is not an f_φ in these lines is the f under the integral.

418^7 A nice follow-up exercise for 18-44 would be the following.

For $k \in \mathbb{N}$, let $f_k(x) := \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{1}{\sqrt{k\pi}} \frac{1}{\left(1+\frac{x^2}{k}\right)^{\frac{k+1}{2}}}$. These are the density functions for Student's t-distributions with k degrees of freedom.

(a) Prove that for every $x \in \mathbb{R}$ we have $\lim_{k\to\infty} \frac{1}{\left(1+\frac{x^2}{k}\right)^{\frac{k+1}{2}}} = e^{-\frac{x^2}{2}}$.

Hint. Use Theorem 12.21.

(b) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Hint. Use Exercise 18-44c and a substitution $x = \frac{v^2}{2}$.

(c) Prove that $\lim_{k\to\infty} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{1}{\sqrt{\frac{k}{2}}} = 1$.

Hint. Distinguish even and odd k, use that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ and use Wallis' product formula.

(d) Prove that $\lim_{k\to\infty} f_k(x) = N_{0,1}(x)$.

Note. This proves that the density of the standard normal distribution is the pointwise limit of the densities of Student's t-distributions with k degrees of freedom for $k \to \infty$.

426_1 (N. Naksinehaboon) Lowercase and capital "f" should be switched to read

$$F(z_1, \ldots, z_d) = f(z_1, \ldots, z_m) + \sum_{j=m+1}^{d} z_j v_j.$$

$428_{13,12}$ The subscripts $\mathbb{R}^d$ for the sets $U_{\mathbb{R}^d}$ and $C_{\mathbb{R}^d}$ would be better if they were subscripts $\mathbb{R}^m$, that is, the sets should better be called $U_{\mathbb{R}^m}$ and $C_{\mathbb{R}^m}$.

439_{11} To be more precise, D_α denotes the partial derivative in the direction of the α^{th} unit vector e_α.

469_8 Piecewise smooth 2π periodic functions have not been defined. Theorem 20.11 should start with "If $f : [-\pi, \pi) \to \mathbb{R}$ is a piecewise smooth function, ..."

469_8 The proof also shows that we have convergence at every $x \in \mathbb{R}$ for which $h(t) := \frac{f(x-t)+f(x+t)-2c}{2\sin\left(\frac{t}{2}\right)} \mathbf{1}_{[0,\pi)}$ with $c := \frac{1}{2}\left[\lim_{u\to x^-} f(u) + \lim_{u\to x^+} f(u)\right]$ is bounded. This means that we get convergence of Fourier series at every point which has a neighborhood in which the derivative is bounded. So the Fourier series of functions like $f(x) = \sqrt{x}$ are guaranteed to converge for all x that are not singularities of the derivative.

475^{18} It may help to add "Consider the real and imaginary parts of f separately." to the hint for Exercise 20-16.

475^{19} Another good exercise here would be the consideration of Fourier series in several variables.

Let $d \in \mathbb{N}$, let $L^2[-\pi, \pi)^d$ be the set of square integrable functions on $[-\pi, \pi)^d$ with the scalar product $\langle f, g\rangle = \frac{1}{\pi^d} \int_{[-\pi,\pi)^d} f(z)g(z)\, dz$ and let

$$\mathcal{T} := \left\{ \prod_{j=1}^{d} f_j(k_j x_j) : f_j(\cdot) \in \left\{\cos(\cdot), \sin(\cdot), \frac{1}{\sqrt{2}}\right\}, k_j \in \mathbb{Z} \right\}$$

be the set of trigonometric polynomials on $[-\pi, \pi)^d$.

(a) Prove that $\mathcal{T}$ is an orthonormal system in $L^2[-\pi, \pi)^d$.

(b) For $j = 1, \dots, d$ let $-\pi \le a_j < b_j < \pi$. Prove that $\overline{\operatorname{span}(\mathcal{T})}$ contains the indicator functions $\mathbf{1}_{\prod_{j=1}^{d}[a_j,b_j]}$.

(c) Prove that $\overline{\operatorname{span}(\mathcal{T})}$ is dense in $L^2[-\pi, \pi)^d$.

(d) Prove that $\mathcal{T}$ is an orthonormal base of $L^2[-\pi, \pi)^d$.

479^{5} It must be added that S is a *surjective* isometry. Surjectivity is the main point in Riesz' Representation Theorem.

515_{3-1} Injectivity can be proved directly as follows.

Let $w \in H$ be so that $T(w) = 0$. Then $0 = \big\langle T(w), w\big\rangle = B(w, w) \ge \lambda\|w\|$, which implies that $w = 0$. Hence T is injective.

523_{4ff} Exercise 23-12 must be rephrased as follows.

Let $\Omega \subseteq \mathbb{R}^d$ be open. Prove that if a function $f \in L^p(\Omega)$ is weakly differentiable, then on every compact box B that is contained in Ω, and on which the weak partial derivatives are in $L^\infty(B)$, f is equal a.e. to a function $g : B \to \mathbb{R}$ that is **absolutely continuous** on B and so that $\frac{\partial g}{\partial x_j} = D^{(j)} f$ a.e. on B° for all $j \in \{1, \dots, d\}$.

Hint. Without loss of generality assume that $B = [0, 1]^d$. Use Fubini's Theorem to prove that for each $j \in \{1, \dots, d\}$ there is a null set $N \subseteq \mathbb{R}^{d-1}$ so that for all $(x_1, \dots, \widehat{x_j}, \dots, x_d) \notin N$ we have

$$\int_0^1 f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)\varphi'(t)\, dt$$
$$= \int_0^1 D^{(j)} f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)\varphi(t)\, dt$$

for φ in a countable dense subset of $C_0^\infty(0,1)$ (so that for every $\psi \in C_0^\infty(0,1)$ there is a sequence with $\varphi_n \to \psi$ and $\varphi_n' \to \psi'$). Then the equation holds for all $\varphi \in C_0^\infty(0,1)$. Conclude

$$f(x_1,\ldots,x_d) = c_j + \int_a^{x_j} D^{(j)} f(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_d)\, d\lambda(t)$$

for almost all $x_j \in [0,1]$ and for all $(x_1,\ldots,\widehat{x_j},\ldots,x_d) \notin N$. Because j was arbitrary, conclude that because the weak first partial derivatives are bounded, f is equal a.e. on B to an absolutely continuous function.

The boundedness of the weak first partial derivatives is needed in the last step. Without some type of uniformity in the absolute continuity of the sections, it is not possible to obtain absolute continuity of the whole function from the absolute continuity of the sections. The function $f(x) := \ln\|x\|$ on the unit ball of $\mathbb{R}^3$ shows that this is not just a problem with the proof, because the L^1 function f is weakly differentiable with weak partial derivatives $\frac{\partial}{\partial x_j}\ln\|x\| = \frac{x_j}{\|x\|^2}$.

528^{16} Because Ω is bounded, $C^\infty\left(\overline{\Omega}\right) \subseteq W^{m,p}(\Omega)$, so there is no need to write $C^\infty\left(\overline{\Omega}\right) \cap W^{m,p}(\Omega)$.

529^{13} Because Ω is bounded, $C^\infty\left(\overline{\Omega}\right) \subseteq W^{m,p}(\Omega)$, so there is no need to write $C^\infty\left(\overline{\Omega}\right) \cap W^{m,p}(\Omega)$.

541_8^7 In Theorems 23.44 and 23.45 it must be demanded that the triangulations are *admissible*.

547^{10} $\{x \in S|P(x)\}$ should read $\{x \in S \mid P(x)\}$

547_1 "$\{a,\{a\}\}$" should read "$a \cup \{a\}$"

555 "determinant" should also be indexed with page 419.

556^4 "discontinuity" should also be indexed with page 301.

559 "normal distribution" and "$N_{\mu,\sigma}$" should be added, indexed with page 418.

561 "standard proof techniques - uniqueness" should be indexed for page 4 not page 3.

562 "volume function" should be added, indexed with page 419.