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Ordered Sets

An Introduction

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Concept Map of the Contents

This text can be read linearly, cover-to-cover as is the case with any good mathematics text. This approach however would delay readers who are interested in the content of the later chapters. As it turns out, such a delay is not necessary.

The concept map in Figure 1 on page x indicates how content can be organized to satisfy a variety of interests. The only common requirement is satisfactory coverage of the (purposely lean) core.

The arrows denote possible straight paths from one segment to another. The connections only indicate some reasonable paths through the text. There may be others, and the indicated paths need not be completely straight (because of interrelations with other topics). Some omitted content "between" the subject blocks may need to be acquired by selective reading at the appropriate place in the later chapter. These gaps should not be a deterrent. For definitions etc. to fill such gaps, please consult the index. Aside from the references in the concept map, the author also highly recommends [19, 26, 218, 220, 224, 227] for further study.

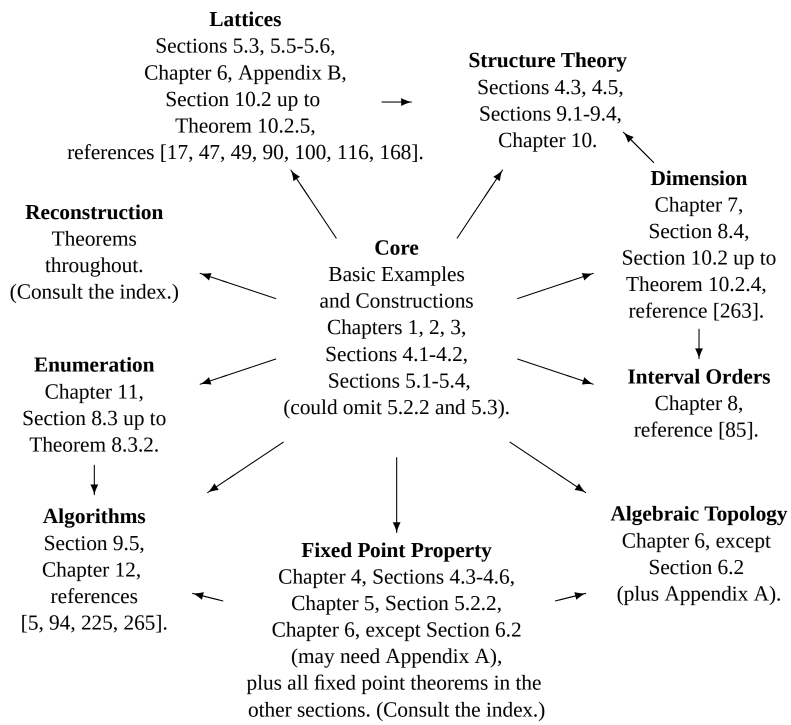

Figure 1: Concept map of the contents. After the common core, the reader can follow any of the arrows to topics of interest without encountering insurmountable gaps.

Preface

Order theory can be seen formally as a subject between lattice theory and graph theory. Indeed, one can say with good reason that lattices are special types of ordered sets, which are in turn special types of directed graphs. Yet this would be much too simplistic an approach. In each theory the distinct strengths and weaknesses of the given structure can be explored. This leads to general as well as discipline-specific questions and results. Of the three research areas mentioned, order theory undoubtedly is the youngest. The first specialist journal *Order* was launched in 1984 and much of the research that guided my own development started in the 1970s.

When I started teaching myself order theory (via a detour through category theory) I was only dimly aware of lattices and graphs. (I was working on a Ph.D. in harmonic analysis and probability theory at the time.) I was attracted to the structure, as it apparently fits the way I think. It was possible to learn the needed basics from research papers as well as surveys. From there it was immensely enjoyable to start working on unexplored problems. This is the beauty of a fresh field. Interesting results are almost asking to be discovered. I hope that the reader will find the same type of attraction to this area (and will ultimately make interesting contributions to the field).

Yet there also is a barrier to entering such a new field. In new fields, standard texts are not yet available. I felt it would be useful to have a text that would expose the reader to order theory as a discipline without too quickly focusing on one specific subarea. In this fashion a broader picture can be seen. This is my attempt at such a text. It contains all that I know about the theory of ordered sets. From here, articles on ordered sets as well as the standard references I had available starting out (which are primarily Rival's Banff conference volumes [218, 220, 225], but also Birkhoff's classic [17] on lattice theory, Fishburn's text [85] on interval orders, and later Davey and Priestley's text [49] on lattices and Trotter's monograph [263] on dimension theory) should be easily accessible. The idea was to describe what I consider the basics of ordered sets without the work becoming totally idiosyncratic. Some of the salient features of this text are bulleted below.

- **Theme-driven approach.** Most of the topics in this text are introduced by investigating how they relate to research problems. We will frequently revisit the open problems that are explained early in the text. Further problems are added as we progress. In this fashion, I believe, the reader will be able to form the necessary intuitions about a new structure more easily than if there were no common undercurrent to the presentation. I have deliberately tried to avoid the often typical beginning of a text in discrete mathematics. There is no deluge of pages upon pages of basic definitions. New notions are introduced when they first arise and they are connected with known ideas as soon as possible.

- **Connections between topics.** Paraphrasing W. Edwards Deming, one can say that "*If we do not understand our work as part of a process, we do not understand our work at all.*" This statement applies to industry, education and also research. Many powerful results have been proved by connecting several seemingly different branches of mathematics. Consequently I tried to show some of the connections between the different areas of ordered sets, as well as connections of the theory of ordered sets to areas such as algebraic topology, analysis, and computer science, to name several. These types of cross-connections have always been fascinating to me. Consider, for example, the use of algebraic topology in Chapter 6. Its connections to ordered sets yield results of a caliber that appears impossible to achieve by staying within a single discipline.

  Along the same lines it must immediately be said that I cannot claim to know all connections between order and other fields. Thus this text is by no means a complete guide to interdisciplinary work that involves order.

- **Breadth and Flexibility.** One of my professors once said that mathematics is a field that one can study for 80 years without repeating a topic, but unfortunately also without contributing anything new. What we don't know will always exceed what we know, and order is no exception. On the other hand, there is much to be said for a broad education. The more one knows, the more connections one can make and the more potential one has to make good contributions.

  To allow for the benefits of broad training without the (very real) drawback of being overloaded with information, the reader may refer to the concept map on p. ix. The concept map shows how sections can be grouped around the core to satisfy a variety of purposes. The core was purposely kept lean.

This organization makes it easier to tackle the breadth of the text and also makes the text more flexible. Follow your strongest interests first, then obtain more information about other things. So, I hope the reader will come back to this text frequently to learn more and to use this knowledge as a springboard towards new work in ordered sets.

If the reader decides to stray from the linear presentation of the text, some results that use examples from earlier sections may need to be skipped (or better, acquired by selective reading).[1] Still, the reader should be able to acquire sufficient understanding of the chapter by relying on work that connects to topics already read. The reader can be guided by his/her primary interests and will still be exposed to many of the cross-connections mentioned above. (This remark should not dissuade the more intrepid readers from going cover-to-cover.)

- **Depth.** The greatest depth one can achieve in any research topic is to understand the open research problems and to be aware of most or all of the results pertaining to their solutions. For at least three problems, the fixed point property, the order reconstruction problem and the automorphism problem, the reader is exposed to essentially all that I know about these topics. I will never claim comprehensive knowledge of a topic. For these problems, however, I am virtually certain that I have gone as far as the theory does to date. More open problems are given in the text as well as in the sections behind the exercises. In this fashion the text allows the reader to attempt research even after reading a few chapters.

- **Open Problems.** The idea to work with a set of open problems certainly is inspired by the early issues of the journal *Order*. Open problems show the frontiers of a field and are thus in some way the life of a branch of mathematics. Therefore I tried to at least show the reader what were and are considered major open problems in ordered sets. The main open problems that are part of the text come from Order's problem list or are inspired by it. I can safely say that they are of interest to the community at large. Naturally the text is biased towards the problems that I know more about. For other problems in the text I have tried to include references to more literature.

  The open problems after the exercise sections and open problem 10.3.7 reflect my own curiosity. These problems range from special cases of the major problems, to simpler proofs of known results, to some things that "I simply would like to know." The only way I have to gauge their difficulty is to say that I have not solved them (yet?). I suspect that some may turn out to be quite difficult (material for MS or Ph.D. theses). I sincerely hope that none of them turn out to be trivial or that their solutions were overlooked by me. Yet this possibility can never be excluded, especially since, to provide

[1] The index, which I tried to make as redundant as possible, should help in filling holes, clarifying unknown notation, etc.

a more complete picture of order, I sometimes stretched myself into areas that I am not as familiar with. Either way, I would be interested in solutions as they arise.

More open problems about order can be found in the collections [218, 220, 225].

- **Standard topics.** There are certain standard topics that mathematicians unfamiliar with order may automatically identify with order theory. The most worked-on parts of the theory of ordered sets appear to be lattice theory and dimension theory. Interval orders also have received a good bit of attention due to their applicability in modeling schedules. There are textbooks available in each of these areas (cf. [17, 49, 85, 263]) and any exposure of these areas as *part* of a text must necessarily be incomplete. Therefore in these areas only the basics and some relations to the main themes of the text are explored. A deeper study of these areas has to be relegated to further specialization. The same can be said for the treatment of constraint satisfaction problems in Chapter 12.

- **Ordering of topics and history.** Like many authors, I want the text to be readable, not a list of achievements in exact chronological order. Thus, whenever necessary, logic supersedes historical order. Historically later results are used freely to prove historically earlier results when this appeared appropriate. The overall presentation is intended to be linear.

  The notes at the end of each chapter (if they refer to historical developments) reflect a very limited historical view at best. I could do no better, as many of the topics this text touches upon (such as reconstruction in general, lattices, dimension theory, constraint satisfaction, to name several) are in fact the tips of rather large icebergs, which in many cases I cannot fully fathom myself.

  The reader who is interested in the history of a subject is advised to look at surveys I have referenced or to run searches of the *Mathematical Reviews* database with the appropriate key word. Electronic search tools are perfect for such tasks. The reader should however be aware that terminology is changing over time.[2]

- **Reading with a pencil** is mandatory, more so than in a regular mathematics text. The diagram of an ordered set is a very strong visual tool. Many proofs that appear difficult at first become clear when drawing diagrams of the described situations. Indeed much of the appeal of the theory of ordered sets derives from its strongly visual character.

[2] In fact, I have spent some time reinventing certain results because I was unaware of the particular vocabulary of an area. While this appears inefficient, the only alternative would be to always just stick with what I know well. This I consider an unacceptable proposition.

Mathematics is learned by doing, by confronting a topic and acquiring the tools to master it. It is time to do so. I hope reading this text will be an enjoyable and intellectually enriching contribution to the reader's mathematical life. Readers interested in code for Chapter 12 or who have solutions to open problems or suggestions for exercises are encouraged to contact the author. I am planning to post updates on my web site.

Ruston, LA, October 24, 2002

– Bernd Schröder

Acknowledgements

This book is dedicated to my family. It has been written largely on their time. Without the support of my wife Claire, my children Samantha, Nicole, Haven and Mlle and of my parents-in-law Merle and Jean, the writing would not have been possible. Moreover, without the help of my parents Gerda and Siegfried and the sacrifices they made I would have never reached the starting point in the first place.

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Finally I would like to thank the readers and the technical staff who helped this book through the final stages.

1
The Basics

There are few prerequisites to this text. The reader should be familiar with real numbers, functions, sets and relations. Moreover the elusive property known as "mathematical maturity" should have been developed to the point that the reader can read and understand proofs and produce simple proofs. A text that develops these skills is for example [103]. A background in graph theory helps, but is not necessary.

1.1 Definition and Examples

This section introduces some of the basic definitions and notations to be used throughout the text. The author has tried to keep the length of this section to a minimum. Terminology is introduced "in context" whenever possible. When encountering unknown notation, terminology, symbols later in the text, the reader should consult the index to locate the definition.

The central concept in this book is the concept of an ordered set, which is a set equipped with a special type of binary relation. Recall that abstractly a binary relation on a set P is just a subset $R \subseteq P \times P = \{(p, q) : p, q \in P\}$. $(p, q) \in R$ simply means that "p is related to q under R". A binary relation R thus contains all the pairs of points that are related to each other under R. For any binary relation R in this text we will write pRq instead of $(p, q) \in R$ whenever p is related to q via R. The relations of most interest to us are order relations.

Definition 1.1.1 *An* **ordered set** *(or* **partially ordered set** *or* **poset***) is an ordered pair* $(P, \leq)$ *of a set* P *and a binary relation* $\leq$ *contained in* $P \times P$*, called the* **order** *(or the* **partial order***) on* P*, such that*

1. *The relation* $\leq$ *is reflexive. That is, each element is related to itself;*
$$\forall p \in P : p \leq p.$$
2. *The relation* $\leq$ *is antisymmetric. That is, if* p *is related to* q *and* q *is related to* p*, then* p *must equal* q*;*
$$\forall p, q \in P : [(p \leq q) \wedge (q \leq p)] \Rightarrow (p = q).$$
3. *The relation* $\leq$ *is transitive. That is, if* p *is related to* q *and* q *is related to* r*, then* p *is related to* r*;*
$$\forall p, q, r \in P : [(p \leq q) \wedge (q \leq r)] \Rightarrow (p \leq r).$$

The elements of P are called the **points** of the ordered set. Order relations introduce a hierarchy on the underlying set. The statement "$p \leq q$" is read "p is less than or equal to q" or "q is greater than or equal to p". The antisymmetry of the order relation ensures that there are no two-way ties ($p \leq q$ and $q \leq p$ for distinct p and q) in the hierarchy. The transitivity (in conjunction with antisymmetry) ensures that no cyclic ties ($p_1 \leq p_2 \leq \cdots \leq p_n \leq p_1$ for distinct $p_1, p_2, \ldots, p_n$) exist. We will also use the notation $q \geq p$ to indicate that $p \leq q$. In keeping with the idea of a hierarchy, we will say that $p, q \in P$ are **comparable** and write $p \sim q$ iff $p \leq q$ or $q \leq p$. We will write $p < q$ for $p \leq q$ and $p \neq q$. In this case we will say p is **(strictly) less than** q An ordered set will be called finite (infinite) iff the underlying set is finite (infinite).

In many disciplines one investigates sets that are equipped with a structure. In cases where no confusion is possible it is customary to not mention the structure explicitly. (Consider for example references like [56] on analysis or [275] on topology.) Since mostly there will be no confusion possible, we will do the same and often not mention the order explicitly. A phrase such as "Let P be an ordered set" normally will mean that P carries an order that is usually denoted by $\leq$. In case we have several orders under consideration, they will be distinguished using subscripts or different symbols. When we have to talk about orders, we will automatically assume that a property of an order is defined in the same way as a property of the ordered set and vice versa.

Example 1.1.2 As we will see in the following sections, ordered structures are present throughout mathematics. To start out, some examples of ordered sets are

1. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ with their usual orders are ordered sets.

2. Any set of sets is ordered by set inclusion $\subseteq$. Similarly, geometric figures (circles in the plane, balls or simplices in higher-dimensional spaces) are ordered by inclusion. Note that if we consider each geometric object as a set of points, we are back to sets ordered by set containment.

3. The simplest example of a set system ordered by inclusion is the **power set** $\mathcal{P}(X)$ of a set X.

4. Every ordinal number in set theory is an ordered set.

5. The vector space $C([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$ can be ordered as follows: For $f, g \in C([0,1],\mathbb{R})$ we say $f \leq g$ iff for all $x \in [0,1]$ we have $f(x) \leq g(x)$ (where the latter $\leq$ is the order of the real numbers).[1]

6. The natural numbers can also be ordered as follows: For $p, q \in \mathbb{N}$ we say $p \sqsubseteq q$ iff p divides q.

7. If J is a set of intervals of the real line, we can order J by

$$[a,b] \leq_{\text{int}} [c,d] \qquad \text{iff} \qquad b \leq c \quad \text{or} \quad [a,b] = [c,d].$$

8. If $(P, \leq_P)$ is an ordered set, the **dual** P^d of P is the set P with the order $\leq_{P^d} := \{(a,b) : b \leq_P a\}$.

9. The lexicographic order on $\mathbb{N}^n$ is defined by $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ iff $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ or there is a $k \in \{1, \ldots, n\}$ with $x_i = y_i$ for $i < k$ and $x_k < y_k$.

Proofs that the structures in Example 1.1.2 truly are ordered sets are not too hard. We will limit ourselves here to proving that the lexicographic order for $\mathbb{N}^n$ is indeed an order.

Proof that Example 1.1.2, part 9 is an ordered set. We will have to prove that $\leq$ is reflexive, antisymmetric and transitive. Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, and $(z_1, \ldots, z_n)$ be elements of $\mathbb{N}^n$.

Reflexivity. By definition we have that $(x_1, \ldots, x_n) \leq (x_1, \ldots, x_n)$.

Antisymmetry. Let

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \quad \text{and} \quad (x_1, \ldots, x_n) \geq (y_1, \ldots, y_n)$$

and assume $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$. Then there are $k_1, k_2 \in \{1, \ldots, n\}$ such that

1. $x_i = y_i$ for $i < k_1$ and $x_{k_1} < y_{k_1}$ and

2. $y_j = x_j$ for $j < k_2$ and $y_{k_2} < x_{k_2}$.

[1] Here it should be clear from the context which order (for numbers or for functions) is meant.

Assume without loss of generality that $k_1 \leq k_2$. Then $x_{k_1} < y_{k_1}$ by 1 and $y_{k_1} \leq x_{k_1}$ by 2, which is a contradiction. Thus we must have

$$(x_1, \dots, x_n) = (y_1, \dots, y_n).$$

This proves antisymmetry.

Transitivity. Let

$$(x_1, \dots, x_n) \leq (y_1, \dots, y_n) \quad \text{and} \quad (y_1, \dots, y_n) \leq (z_1, \dots, z_n).$$

If any two of the three tuples are equal, then there is nothing to prove. Hence we can assume the three tuples are mutually distinct. In this case there are $k_1, k_2 \in \{1, \dots, n\}$ such that

1. $x_i = y_i$ for $i < k_1$ and $x_{k_1} < y_{k_1}$, and
2. $y_j = z_j$ for $j < k_2$ and $y_{k_2} < z_{k_2}$.

Let $k := \min\{k_1, k_2\}$. Then for $i < k$ we have $x_i = y_i = z_i$ and for the index k we have $x_k \leq y_k \leq z_k$ with at least one of the inequalities being strict. Thus $x_k < z_k$ and $(x_1, \dots, x_n) \leq (z_1, \dots, z_n)$, which concludes our proof of transitivity. ∎

It is a good exercise for the reader to prove that every example in 1.1.2 is an ordered set. Not every relation that looks like it induces a hierarchy is an order relation however.

Proposition 1.1.3 (Cf. [270].) *If $\mathcal{J}$ is a set of subsets of an ordered set P, we can define $A \sqsubseteq B$ iff*

1. *For all $a \in A$ there is an element b in B such that $a \leq b$, and*
2. *For all $b \in B$ there is an element a in A such that $b \geq a$,*

Then $\sqsubseteq$ is reflexive and transitive, but not necessarily antisymmetric.

Proof. Let P be an ordered set, $\mathcal{J}$ a set of subsets of P and let $A, B, C \in \mathcal{J}$ be arbitrary. Since for all $a \in A$ we have $a \leq a$, we obtain $A \sqsubseteq A$ for all $A \in \mathcal{J}$. Thus $\sqsubseteq$ is reflexive.

If $A \sqsubseteq B \sqsubseteq C$, then for all $a \in A$ there is a $b \in B$ with $a \leq b$. For b there is a $c \in C$ such that $b \leq c$ and hence $a \leq c$. Similarly, for all $c \in C$ there is a $b \in B$ with $c \geq b$. For b there is an $a \in A$ such that $b \geq a$ and hence $c \geq a$. Thus $A \sqsubseteq C$ and $\sqsubseteq$ is transitive.

However $\sqsubseteq$ is in general not antisymmetric. Consider the sets $\{1, 2, 4\}$ and $\{1, 3, 4\}$ as subsets of $\mathbb{N}$ with the natural order. Then $\{1, 2, 4\} \sqsubseteq \{1, 3, 4\}$ and $\{1, 2, 4\} \sqsupseteq \{1, 3, 4\}$, but clearly the two sets are not equal. ∎

1.2 The Diagram

Having defined ordered sets as sets equipped with a certain type of relation, we are ready to investigate these entities. Yet it would be helpful to have a visual aid to work with ordered sets. A picture often says more than a thousand words. Such a visual aid is inspired by graph theory, so let us quickly review what a graph is.

Definition 1.2.1 *A* **graph** *G is a pair (V, E) of a set V (of* **vertices***) and a set $E \subseteq \{\{a, b\} : a, b \in V\}$ (of* **edges***).*

This is a perfectly fine definition. However when working with graphs most people think not of the set theoretical entities of Definition 1.2.1. Instead they visualize an entity such as shown in Figure 1.1 a). The connection is simple: For each vertex $v \in V$ we put a point in the plane (or in 3-space) to represent the vertex. For any two vertices $v, w \in V$ we join the corresponding points with a line (an edge) that does not touch any other points iff $\{v, w\} \in E$. In this fashion we have a good visual tool for the work with graphs and also a way to translate real life problems into mathematics (road networks for example can be modeled using graphs). We could now do the same thing for orders. Put points in the plane or 3-space and join related points with edges. Arrows could indicate the way in which points are related. This idea would have two shortcomings. First, the hierarchical structure of the order may be hard to detect and second, there will be many lines that can be considered superfluous because of transitivity. We shall tackle the second problem first.

Definition 1.2.2 *Let P be an ordered set. Then $p \in P$ is called a* **lower cover** *of $q \in P$ (and q is called an* **upper cover** *of p) iff $p < q$ and for all $z \in P$ we have that $p \leq z \leq q$ implies $z \in \{p, q\}$. In this case we write $p \prec q$. Points p and q that satisfy $p \prec q$ or $q \prec p$ will also be called* **adjacent***.*

Example 1.2.3 To become familiar with the covering relation, consider.

1. In the power set $\mathcal{P}(\{1, \ldots, 6\})$ the set $\{1, 3\}$ is a lower cover of $\{1, 3, 5\}$, but it is not a lower cover of $\{1, 2, 3, 4\}$.

2. In $\mathbb{Z}$ each number k has exactly one upper cover $(k+1)$ and one lower cover $(k-1)$.

3. Whether two elements are covers of each other depends on the surrounding universe. 2 is not an upper cover of 0 in $\mathbb{Z}$, but it is an upper cover of zero in the set of even numbers. Similarly $\{1, 2, 3, 4\}$ is an upper cover of $\{1, 3\}$ in the set of subsets of $\{1, \ldots, 6\}$ that have an even number of elements.

4. In infinite ordered sets, elements may or may not have covers. Consider that in $\mathbb{R}$ and $\mathbb{Q}$ no two elements are covers of each other.

The covering relation carries no superfluous information. It is the smallest relation that carries all the information for a given finite order. To visually incorporate the hierarchy, we only have to impose an up-down direction. The resulting tool that is analogous to the sketch of a graph is the Hasse diagram. Its main use is, because of the difficulty indicated in Example 1.2.3, part 4, for finite ordered sets.

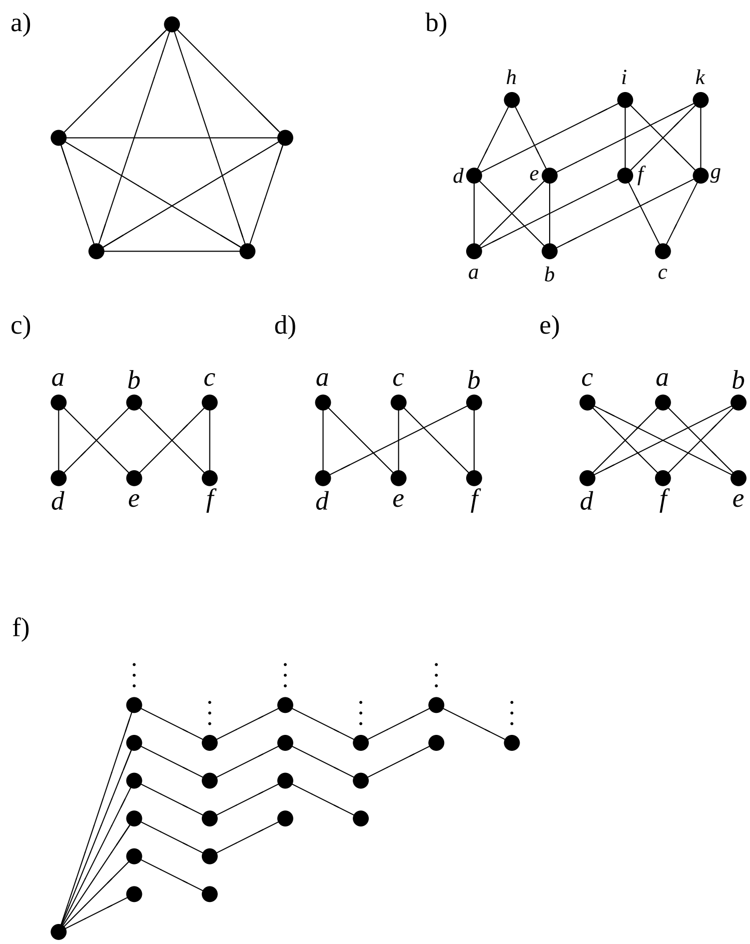

Figure 1.1: a) The complete graph with five vertices K_5, b) The diagram of an ordered set (the set $P2$ in [235]), c)–e) Three diagrams for the same ordered set (points to be identified by an isomorphism have the same letters), f) The "diagram" of an infinite ordered set (this "spider", as Rutkowski named it, is taken from [270], Remark 5.1).

Definition 1.2.4 *The* **(Hasse) diagram**[2] *of a finite ordered set* P *is the ordered pair* $(P, \prec)$*, where* $\prec$ *is the lower cover relation as defined in Definition 1.2.2.*

Again a perfectly reasonable mathematical definition which gives rise to the following possibility for visualization (cf. Figure 1.1, b-e).

1. Draw the points of P in the plane (or in 3-space, or on some surface) such that if $p \leq q$, then the point for q has a larger y-coordinate (z-coordinate) than the point for p,
2. If $p \prec q$, draw a line or curve (an edge) between the points corresponding to p and q such that
   - The slope of the edge (or $\frac{\partial}{\partial z}$ if we are in 3-space and just assume that our connecting curves are somewhat smooth) does not change its sign and
   - The edge does not touch any points of P except those corresponding to p and q.

   Do not join any other pairs of points by edges.

For purpose of illustration consider Figure 1.1, part b). We can easily read off the comparabilities in the diagram shown. For example, $a \leq d$, since there is an edge from a to d and a is lower than d. We also have $a \leq k$, since there is an edge going up from a to e and another going up from e to k. This means $a \prec e \prec k$ and by transitivity $a \leq k$. The knowledge that order relations are transitive allows us to avoid drawing some edges that would only clutter the picture. Just imagine all the edges (a, h), (a, i), (a, k), (b, h), (b, i), (b, k), (c, i), and (c, k) added to the picture. It would be quite confusing. Also note that from Figure 1.1 part b) we see that $c \not\leq h$. Indeed, even though there is an edge from c to f, an edge from f to a, an edge from a to d and an edge from d to h, not all the edges are traversed in an upward direction in the trail just described. The edge from f to a is traversed downwards (from f to a), meaning that $c \prec f \succ a \prec d \prec h$. Transitivity cannot be applied to this sequence (and not to any other such sequence) and so $c \not\leq h$.

Drawing diagrams is not canonical. The same ordered set can be drawn in different ways according to the investigator's preferences or needs. As an example, consider Figure 1.1, parts c)–e). Each picture depicts the same ordered set, yet they do look distinctly different.

Diagrams can also be drawn for infinite ordered sets, but are then in need of explanation. The infinite ordered set depicted in Figure 1.1 f) consists of one "zig-zag" with n elements for each $n \in \mathbb{N}$ such that all "zig-zags" have the same left endpoint. How to draw a diagram of a certain set is often a matter of taste. For some discussion on the subject cf. [4]. For drawing diagrams on surfaces cf. [77]

[2] Named after the German algebraic number theorist Helmut Hasse who used diagrams to picture the ordered sets of subfields or field extensions.

and for a multitude of results regarding diagrams cf. [221]. We will use diagrams extensively as visual tools. Drawings of diagrams in this text are biased towards the author's taste.

From a relation-theoretic point-of-view the diagram is a certain subset of the order relation, formed according to the rule that only pairs (a, b) are selected for which there is no intermediate point i such that (a, i) and (i, b) are also in the relation. To formalize how to recover the original relation from the diagram we merely need to formulate the reading process indicated above as a mathematical construction.

Definition 1.2.5 *Let $\prec$ be a binary relation on the finite set P. Then the* **transitive closure** $\prec^t$ *of $\prec$ is defined by $a \prec^t b$ iff there is a sequence $a = a_1, a_2, \ldots, a_n = b$ such that $a_1 \prec a_2 \prec \cdots \prec a_n$.*

The name is justified by the following result and by Exercise 7.

Proposition 1.2.6 *The transitive closure of a binary relation $\prec$ on a finite set P is transitive.*

Proof. Let $a, b, c \in P$ with $a \prec^t b \prec^t c$. There are $a = a_1, a_2, \ldots, a_n = b$ with $a_1 \prec a_2 \prec \cdots \prec a_n$ and $b = a_n, a_{n+1}, \ldots, a_{n+m} = c$ with $a_n \prec a_{n+1} \prec \cdots \prec a_{n+m}$. Therefore $a = a_1 \prec a_2 \prec \cdots \prec a_{n+m} = c$ and hence $a \prec^t c$. ■

For finite sets, the transitive closure allows us to translate back from the diagram to the order relation.

Proposition 1.2.7 *Let P be a finite ordered set with order $\leq$ and let $\prec$ be its lower cover relation. As usual, $=$ denotes the equality relation. Then $\leq$ is the transitive closure of the union of the relations $\prec$ and $=$.*

Proof. Let $\leq'$ denote the transitive closure of $\prec \cup =$. (Note that despite the suggestive notation, at this stage we do not even know if $\leq'$ is an order relation.) We will prove that $\leq = \leq'$. Since $\prec \cup =$ is contained in $\leq$ and since $\leq$ is transitive, we must have that $\leq' \subseteq \leq$ (cf. Exercise 7).

To prove the other inclusion, assume there are $a, b \in P$ with $a \leq b$ and $a \not\leq' b$. Then we must have $a < b$, since $a = b$ implies $a \leq' b$. Since P is finite we can find points $c, d \in P$ with $c \leq d$ and $c \not\leq' d$ such that for all $c < z < d$ we have $c \leq' z$ and $z \leq' d$. Since $c \prec d$ would mean $c \leq' d$, there must be a $z \in P$ with $c < z < d$. But then $c \leq' z \leq' d$ and since the transitive closure of a relation is transitive, we infer $c \leq' d$, a contradiction. Thus a, b as described above cannot exist and we conclude that $\leq = \leq'$. ■

We will discuss some algorithmic ramifications of computing the diagram and transitive closures in Sections 12.1 and 12.2.

1.3 Order-Preserving Mappings/Isomorphism

Figure 1.1 c), d) and e) shows three different pictures of the same ordered set. Had we assigned different labels to the points, we could have depicted three ordered sets that appear "different and yet the same". This phenomenon is similar to the topological result (which seems to have become folklore), that "a donut is homotopic/isomorphic to a teacup". Indeed we can find a "continuous deformation" that has diagrams at each stage and, say, turns set c) into set d). The formal background lies in the investigation of structure-preserving maps (or "morphisms") which is strongly represented in algebra and topology (cf. the strong role of structure homomorphisms in algebra and of continuous functions in topology). Since the underlying structure we are interested in is the order, the following definition is only natural.

Definition 1.3.1 *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be ordered sets and let $f : P \to Q$ be a map. Then f is called an* **order-preserving function** *iff for all $p_1, p_2 \in P$ we have*

$$p_1 \leq_P p_2 \;\Rightarrow\; f(p_1) \leq_Q f(p_2).$$

(We will usually not index the orders in such a situation. Also we will use the words "function" and "map" interchangeably. Finally let it be noted that order-preserving maps are sometimes also referred to as **isotone maps***.)*

Example 1.3.2 We continue with some examples of order-preserving and non-order-preserving maps. Note that the same map can be order-preserving or not, depending on the orders of domain and range.

1. The function $f : \mathbb{N} \to \mathbb{N}$, defined by $f(x) = 5x$ is an order-preserving map if both domain and range carry the natural order. It is also order-preserving if both domain and range carry the order $\sqsubseteq$ of part 6 in Example 1.1.2.

2. The function $f : \mathbb{N} \to \mathbb{N}$, defined by $f(x) = x + 1$, is order-preserving if both domain and range carry the natural order. However, it is not order-preserving if both domain and range carry the order $\sqsubseteq$ of part 6 in Example 1.1.2.

3. Let S be a set and let $\mathcal{P}(S)$ be its power set ordered by set inclusion. The map $\mathcal{I} : \mathcal{P}(S) \to \mathcal{P}(S)^d$ defined by $\mathcal{I}(X) := S \setminus X$ is an order-preserving map from $\mathcal{P}(S)$ to its dual.

4. Let B be the ordered set in part b) of Figure 1.1 and let C be the ordered set in part c) of Figure 1.1. Then the function $F : B \to C$ defined by $F(h) = a$, $F(a) = F(b) = F(d) = F(e) = d$, $F(f) = F(g) = F(i) = F(k) = b$, and $F(c) = f$ is order-preserving. We use reflexivity of order relations here. This property (though essentially taken for granted and not noted on the diagram) allows us to collapse related points into one.

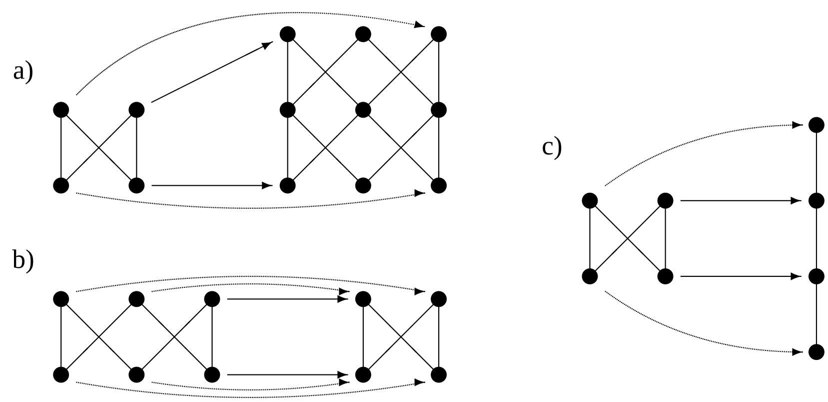

Figure 1.2: Graphical representation of some order-preserving maps.

5. Consider Figure 1.2. The sets where the arrows start are the domains of the maps. The arrows indicate where each individual point is mapped. All the maps thus given in Figure 1.2 are order-preserving.

If Definition 1.3.1 is a reasonable definition for structure-preserving maps on ordered sets, then the composition of two order-preserving maps should again be order-preserving.

Proposition 1.3.3 *Let P, Q, R be ordered sets and let $f : P \to Q$ and $g : Q \to R$ be order-preserving maps. Then the map $g \circ f : P \to R$ is also order-preserving,*

Proof. Let $p_1, p_2 \in P$ with $p_1 \leq p_2$. Then since f is order-preserving we have $f(p_1) \leq f(p_2)$ and since g is order-preserving $g(f(p_1)) \leq g(f(p_2))$. Thus $g \circ f$ is order-preserving. ■

The above examples of order-preserving maps show that these maps preserve the order "one way". However, even the existence of a bijective order-preserving map between two sets, such as in part c) of Figure 1.2, is not a guarantee that both sets are "essentially the same". (This is similar to the situation in algebra or topology.) What is missing in part c) of Figure 1.2 is that the inverse function is not order-preserving.

Suppose now the two ordered sets P and Q have a bijection Φ between them such that Φ as well as Φ^{-1} preserve the order. When investigating an ordered structure, the underlying set normally only gives us the substance that we mold into structures. Which individual point is placed where in the structure is thus quite unimportant. Hence for many purposes in order theory, P and Q as above are indistinguishable, as they have the same order-theoretical structure. This is the concept of (order-)isomorphism.

Definition 1.3.4 *Let P, Q be ordered sets and let $\Phi : P \to Q$. Then Φ is called an* **(order-)isomorphism** *iff*

1. *Φ is order-preserving,*
2. *Φ has an inverse Φ^{-1},*
3. *Φ^{-1} is order-preserving.*

The ordered sets P and Q are called **(order-)isomorphic** *iff there is an isomorphism $\Phi : P \to Q$.*

The following characterization of isomorphisms reinforces the notion that isomorphic ordered sets can be regarded as "the same".

Proposition 1.3.5 *Let P, Q be ordered sets. Then $f : P \to Q$ is an order-isomorphism iff*

1. *f is bijective.*
2. *For all $p_1, p_2 \in P$ we have*

$$p_1 \leq p_2 \ \Leftrightarrow \ f(p_1) \leq f(p_2).$$

Proof. If $f : P \to Q$ is an order-isomorphism, then 1 and 2 are trivial. To prove that 1 and 2 imply f is an isomorphism, we only need to prove that $f^{-1} : Q \to P$ is order-preserving. Let $q_1, q_2 \in Q$ be such that $q_1 \leq q_2$. Then there are $p_1, p_2 \in P$ such that $f(p_i) = q_i$ for $i = 1, 2$. Now $f(p_1) = q_1 \leq q_2 = f(p_2)$ implies $p_1 \leq p_2$, that is, $f^{-1}(q_1) \leq f^{-1}(q_2)$. Thus f^{-1} is order-preserving. ■

Proposition 1.3.6 *Let P, Q, R be ordered sets and let $\Phi : P \to Q$ and $\Psi : Q \to R$ be order-isomorphisms. Then $\Psi \circ \Phi$ is an order-isomorphism.*

Proof. Left as Exercise 14. ■

The strength of using isomorphisms is that structures that at first appear different can turn out to be equal for all intents and purposes. Thus even structures that appear different can have the same properties. For example, the ordered sets in Figure 1.3 appear quite different at first, yet they are isomorphic via the map (from the left set to the right set) $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 5$, $6 \mapsto 8$, $7 \mapsto 6$, $8 \mapsto 7$, $9 \mapsto 10$, $10 \mapsto 9$, $11 \mapsto 11$. Verification that the indicated map is an isomorphism is a good exercise for the reader. While the experienced reader can (and should) classify this task as "tedious, but routine", it gives an indication how hard it is to find an isomorphism. How many calculations would have been necessary to find the indicated map, had it not been given above? Even worse, what if we start out with sets of which we do not know if they are isomorphic

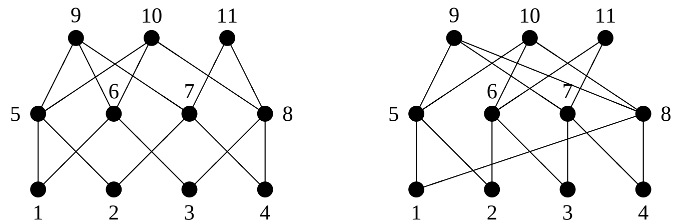

Figure 1.3: Two isomorphic ordered sets.

or not? How do we know how many and what types of checks to perform until we can be sure no isomorphism exists? These questions show that it can be quite difficult to decide if two ordered sets are isomorphic or not. In fact it is still an open problem *how* difficult this decision is. We will discuss this issue in Remark 8 in Chapter 12.

1.4 Fixed Points

A property that has attracted a good bit of attention in order theory is the fixed point property, defined as follows.

Definition 1.4.1 *We will call a function $f : D \to D$ whose domain and range are equal a* **self map** *on D. $f : D \to D$ is an* **order-preserving self map** *on D iff D carries an order $\leq$ and $a \leq b$ implies $f(a) \leq f(b)$.*

Definition 1.4.2 *Let P be an ordered set and let $f : P \to P$ be an order-preserving self map. Then $p \in P$ is called a* **fixed point** *of f iff $f(p) = p$. If f has no fixed points, f is called* **fixed point free**. *P is said to have the* **fixed point property** *iff each order-preserving self map $f : P \to P$ has a fixed point. For any ordered set P and any order-preserving map $f : P \to P$ we set*

$$\text{Fix}(f) := \{p \in P : f(p) = p\}.$$

An original motivation for working with fixed points in ordered sets is a proof of the Bernstein–Cantor–Schröder theorem, which we will give later in the text. Yet the property also became interesting in itself. For more background on the fixed point property, consider Remark 2 in the "Remarks and Open Problems" section of this chapter.

We will use the fixed point property as a vehicle to introduce the reader to new order-theoretical notions. The fixed point property is well-suited for this task, because it combines properties of the set with properties of the maps on the set. For every new structure we introduce, the fixed point property can provide a familiar setting in which to investigate the structure. The underlying problem is

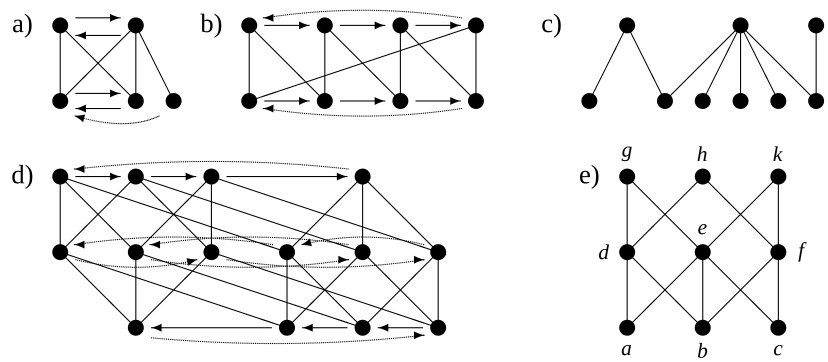

Figure 1.4: Examples of ordered sets with and without the fixed point property. Fixed point free maps are indicated for sets without the fixed point property.

Open Question 1.4.3 (Cf. [223].) *Characterize those (finite) ordered sets that have the fixed point property.*

We will frequently prove fixed point results in this text to show how one can work with a certain class of ordered sets or a certain structure. Ultimately (cf. Theorem 12.4.5) we will prove that the NP version of this problem turns out to be NP-complete.[3] This will lead into our discussion of methods to tackle "hard" algorithmic problems. At this stage we only give a few examples of sets with and without the fixed point property (cf. Figure 1.4).

For the ordered sets in Figure 1.4 that do not have the fixed point property, a fixed point free order-preserving self map is indicated. The question is how to prove that such a map does *not* exist. It is certainly instructive to try and do this for the remaining sets before reading on. We will soon have more sophisticated methods to prove that the ordered sets in Figure 1.4 c) and e) have the fixed point property. To get the reader more acquainted with the basic notions of order theory we give a quick proof that the ordered set in 1.4 e) has the fixed point property. The proof will be combinatorial with potentially many cases to be treated separately. To reduce the number of cases to be treated, one can often use symmetry or, formally, the notion of an automorphism.

Definition 1.4.4 *Let P be an ordered set. The self map $f : P \to P$ is called an* **(order-)automorphism** *iff f is an isomorphism.*

Example 1.4.5 For the ordered set in Figure 1.4 e) the map Φ that maps $a \mapsto c$, $b \mapsto b$, $c \mapsto a$, $d \mapsto f$, $e \mapsto e$, $f \mapsto d$, $g \mapsto k$, $h \mapsto h$, $k \mapsto g$ is an automorphism. We can see that this map "reflects the ordered set across an axis

[3]Formally, deciding if a finite ordered set has the fixed point property is co-NP-complete.

through the middle" if the set is drawn as in Figure 1.4. This illustrates the common interpretation that automorphisms reveal the symmetries of a combinatorial structure.

Proposition 1.4.6 *The ordered set in Figure 1.4 e), call it P, has the fixed point property.*

Proof. Assume to the contrary that there is an order-preserving map $F : P \to P$ such that F has no fixed point. Then $F(b)$ cannot be related to b. Indeed, otherwise $b < F(b)$ and applying F twice to this inequality leads to (since F has no fixed points) $F(b) < F^2(b)$ and $F^2(b) < F^3(b)$. This however is not possible, since there are no four distinct elements $w, x, y, z \in P$ such that $w < x < y < z$. Thus $F(b) \in \{a, c\}$. Since Φ in Example 1.4.5 is an automorphism, we can assume without loss of generality that $F(b) = a$. (Otherwise we would apply the whole following argument to $\Phi^{-1} \circ F \circ \Phi$, which is also a fixed-point-free order-preserving self map.)

Since $F(b) = a$ we have $F[\{d, e, g, h, k\}] \subseteq \{a, d, e, g, h, k\}$. Since $F(g)$ cannot be related to g (proved similar to $b \not\sim F(b)$) we must have $F(g) \in \{h, k\}$. If $F(g) = h$, then we must have that $a = F(b) \leq F(d) \leq F(g) = h$.

$F(d) = h$ would lead to $F(h) \geq F(d) = h$ and then $F(h) = h$, which is not possible. We exclude $F(d) = a$ in similar manner. This leaves $F(d) = d$, a contradiction.

Therefore we must have $F(g) = k$, which then can be lead to a contradiction in similar fashion. Thus P has no fixed-point-free order-preserving self maps and hence P has the fixed point property. ■

With automorphisms being indicators of symmetry, ordered sets that have automorphisms without any fixed points should have a very high degree of symmetry, since there will be at least one way to move every point and still have the same order-theoretical structure. Existence of fixed point free automorphisms is also a problem similar to asking about the existence of a fixed point free order-preserving self map. We record

Definition 1.4.7 *Let P be an ordered set. P is called* **automorphic** *iff P has a fixed point free automorphism.*

Open Question 1.4.8 *Characterize the (finite) automorphic ordered sets.*

Automorphic ordered sets play a role in the investigation of the fixed point property as we will see in Theorem 4.2.2, part 2. The NP version of problem 1.4.8 turns out to be NP-complete as well (cf. [277]). We conclude this section by showing that while the fixed point property implies the ordered set is not automorphic, nonautomorphic ordered sets need not have the fixed point property.

Proposition 1.4.9 *The ordered set in Figure 1.4 a), call it Q, is not automorphic. However, it does not have the fixed point property.*

Proof. A fixed point free order-preserving map is indicated in the figure. To see that Q has no fixed point free automorphism, note that Q has exactly one point with exactly one upper cover. This point must be fixed by any automorphism. ■

1.5 Ordered Subsets/The Reconstruction Problem

Having introduced the objects of our studies and their structure-preserving maps, the last basic notion to expose the reader to are the subobjects. These subobjects are the ordered subsets of an ordered set, which are defined analogous to subgroups in algebra or topological subspaces in topology.

Proposition 1.5.1 *Let $(P, \leq_P)$ be an ordered set and let $Q \subseteq P$. Then Q with the restriction $\leq_Q := \leq_P |_{Q\times Q}$ of the order on P to Q is also an ordered set.*

Proof. The proofs of reflexivity, antisymmetry and transitivity of $\leq_Q$ are trivial. Every property that holds for all elements of P will clearly hold for all elements of Q. ■

Knowing that the order properties are not destroyed when restricting ourselves to a subset, the following definition is sound.

Definition 1.5.2 *Let $(P, \leq_P)$ be an ordered set and let $Q \subseteq P$. If $Q \subseteq P$ and $\leq_Q = \leq_P |_{Q\times Q}$ we will call $(Q, \leq_Q)$ an* **ordered subset** *(or* **subposet***) of P. Unless indicated otherwise we will always assume that subsets of ordered sets carry the order induced by the surrounding ordered set.*[4]

Example 1.5.3 Every set of sets $\mathcal{S}$ that is ordered by inclusion is an ordered subset of $\mathcal{P}\left(\bigcup \mathcal{S}\right)$.

Order-theoretical properties may or may not carry over to ordered subsets. As we have not explored many properties yet, all we can do is record a negative example.

Example 1.5.4 Not every subset of an ordered set with the fixed point property again has the fixed point property. Indeed the subset $\{a, c, g, k\}$ of the ordered set in Figure 1.4 e) does not have the fixed point property. A fixed point free order-preserving map on $\{a, c, g, k\}$ would be $a \mapsto b$, $b \mapsto a$, $g \mapsto k$, $k \mapsto g$.

Connecting ordered subsets with order-preserving functions is simple, since of course for each $f : P \to Q$ the set $f[P]$ is an ordered subset of Q. Moreover it

[4] For those readers acquainted with graph theory this means that the notion of ordered subset is similar to the notion of an *induced* subgraph.

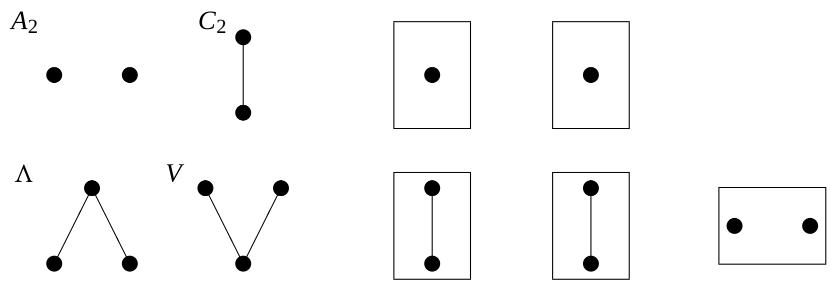

Figure 1.5: Two examples of nonisomorphic ordered sets with isomorphic decks. Are these the only ones?

is easy to see that, if P is finite and f is injective, then $f[P]$ has at least as many comparabilities as P. However, Figure 1.2 part c) shows that $f[P]$ need not be isomorphic to P, even if f is bijective. Order-preserving mappings for which P is isomorphic to $f[P]$ are called embeddings.

Definition 1.5.5 *Let P, Q be ordered sets. Then $f : P \to Q$ is called an* **(order) embedding** *iff*

1. *f is injective.*
2. *For all $p_1, p_2 \in P$ we have*

$$p_1 \leq p_2 \;\Leftrightarrow\; f(p_1) \leq f(p_2).$$

Proposition 1.5.6 *Let P be an ordered set and let $f : P \to Q$ be order-preserving. Then P is isomorphic to $f[P]$ iff f is an embedding. In this case f is an isomorphism between P and $f[P]$.*

Proof. This follows immediately from Definition 1.5.5 and Proposition 1.3.5. ■

To further illustrate ordered subsets we are now ready to introduce our second main research problem (characterizing the fixed point property was the first). Like the fixed point property, this problem will provide us with a familiar setting in which to investigate many new structures we introduce. Draw each subset of an ordered set P that has one point less than P on a card (without labeling the points). Is it possible to take these pictures and reconstruct the original ordered set from them up to isomorphism? Examples that the reconstruction does not work in general are shown in Figure 1.5. However these are the only examples known so far. The natural question that arises is now if these are all such examples.

Definition 1.5.7 *For an ordered set P, we call the class of all ordered sets that are isomorphic to P the* **isomorphism class** *of P. We denote the isomorphism class of P by $[P]$.*

Definition 1.5.8 *Let P be a finite ordered set. For $x \in P$, the ordered subset $P \setminus \{x\}$ is called a* **card** *of P. Cards are generally considered unlabeled. That is, we have no way of determining which element of the card corresponds to which element of P.*

Let $\mathcal{C}$ be the set of all isomorphism classes of ordered sets with underlying set contained in $\mathbb{N}$. The **deck** *of P is the function $\mathcal{D}_P : \mathcal{C} \to \mathbb{N}$ such that for each $[C] \in \mathcal{C}$ we have that $\mathcal{D}_P([C])$ is the number of cards of P that are isomorphic to the elements of $[C]$.*

Open Question 1.5.9 The Reconstruction Problem. *Is every (finite) ordered set with more than three elements uniquely reconstructible from its deck? That is, is it true that if P, Q are ordered sets with more than three elements such that $\mathcal{D}_P = \mathcal{D}_Q$, then P and Q must be isomorphic?*

For more background on the reconstruction problem consider Remark 4 in the "Remarks and Open Problems" section of this chapter. We conclude this section with two simple-looking results which will be helpful when working on reconstruction problems. The first (Proposition 1.5.11) is a result about a partial success by proving a positive answer to the problem in a restricted class of ordered sets. The second (Proposition 1.5.14) shows that a certain parameter can be reconstructed from the deck of any ordered set. These results provide some (of the many) indications the answer might be positive in general. They will also be tools in later investigations. Both results are representative of possible approaches to this problem. One could prove reconstructibility for more and more special classes of ordered sets until every ordered set must belong to one class that has been proved to be reconstructible. Or, one could reconstruct more and more parameters of ordered sets until every ordered set is uniquely determined just by knowing a set of reconstructible parameters. So far, we are far from either of these goals.

Definition 1.5.10 *We will say an ordered set P is* **reconstructible** *from its deck $\mathcal{D}_P$, if all ordered sets Q with $\mathcal{D}_P = \mathcal{D}_Q$ are isomorphic to P. A class $\mathcal{K}$ of ordered sets is reconstructible iff each of its members $P \in \mathcal{K}$ is reconstructible. We will call a class $\mathcal{K}$ of ordered sets* **recognizable** *iff for each ordered set $P \in \mathcal{K}$ and any ordered set Q, $\mathcal{D}_P = \mathcal{D}_Q$ implies that $Q \in \mathcal{K}$.*

Proposition 1.5.11 *Let P be a finite ordered set with $|P| \geq 4$ and a smallest element s. That is, $s \in P$ is such that for all $p \in P$ we have $s \leq p$. Then P is reconstructible from its deck.*

Proof. We will first prove that ordered sets with a smallest element are recognizable. Indeed, if P has a smallest element s, then for any $x \in P \setminus \{s\}$ we have that s is the smallest element of $P \setminus \{x\}$. Thus there is at most one card of P that does not have a smallest element. On the other hand, if Q does not have a smallest element, then there are at least two incomparable elements in Q that do not have any strict lower bounds. Such sets Q have at most two cards that have a smallest element and hence at least two cards that do not. (Cards are counted with multiplicity here.) Thus the ordered sets with a smallest element are exactly those ordered sets whose deck has at most one card without a smallest element.

Now we are ready to prove reconstructibility. If P has a smallest element and the deck of P has a card C that does not have a smallest element, then this card must by the above be unique. Thus $C = P \setminus \{s\}$ and P is isomorphic to the ordered set obtained by attaching a smallest element to C. Formally, P is isomorphic to $C \cup \{s\}$ ordered by $\leq \cup \{(s, c) : c \in C \cup \{s\}\}$, where $\leq$ is the order on C.

If all cards of P have a smallest element, then removal of s must have introduced a new smallest element and we argue as follows. For every finite ordered set Q with a smallest element there is a number l_Q such that

1. All elements of Q with fewer than l_Q lower bounds are unique.

2. Q contains zero or at least two elements that have l_Q lower bounds.

Essentially, $l_Q - 1$ is the number of times a smallest element can be removed before we arrive at an ordered set without a smallest element. For each card C of P find the number l_C. The cards C with $l_C \leq l_K$ for all cards K of P are all isomorphic. Moreover, $P \setminus \{s\}$ is isomorphic to one of them. Pick one and call it C. Then P is isomorphic to the ordered set obtained from C by attaching a new smallest element below the smallest element of C. ■

Definition 1.5.12 *An* **(order) invariant** $\alpha(\cdot)$ *of ordered sets is a function from the class of all ordered sets to another set or class, such that if P and Q are isomorphic, then $\alpha(P) = \alpha(Q)$.*

Invariants are mostly numerical, but the deck for example is also an invariant. The simplest invariant is probably the number of elements, closely followed by the number of comparabilities.

Definition 1.5.13 *An invariant α is called* **reconstructible** *iff for all ordered sets P and Q we have that $\mathcal{D}_P = \mathcal{D}_Q$ implies $\alpha(P) = \alpha(Q)$.*

The most easily reconstructed invariant is the number of elements. It is simply one more than the number of elements of any card. The number of comparabilities is to be reconstructed in Exercise 25a. If one could reconstruct a complete set of invariants, i.e., a set of invariants so that two ordered sets with the same invariants must be isomorphic, then the reconstruction problem would be solved. Unfortunately no such complete set of invariants has materialized yet. A helpful invariant is the number of subsets of a certain type.

Proposition 1.5.14 (A Kelly Lemma, cf. [136], Lemma 1 and [151], Lemma 4.1.) *Let P and Q be two finite ordered sets with $|P| > |Q|$ and $|P| > 3$. Then the number $s(Q, P)$ of ordered subsets of P that are isomorphic to Q is reconstructible from the deck.*

Proof. Let $d_Q := \sum_C |\{S \subseteq C : S \text{ isomorphic to } Q\}|$, where the sum runs over all cards C of P, with multiplicity. Clearly d_Q can be computed from the deck. Let $P_Q \subseteq P$ be a fixed subset of P that is isomorphic to Q. Then P_Q is contained in exactly $|P| - |Q|$ cards of P, namely in exactly those cards obtained by removing an element outside P_Q. Since this is true for any subset of P that is isomorphic to Q, we have that

$$d_Q = s(Q, P)(|P| - |Q|)$$

and we have reconstructed $s(Q, P)$ as

$$s(Q, P) = \frac{d_Q}{(|P| - |Q|)}.$$

■

Exercises

1. Prove that the following are ordered sets:

   (a) The set $\{a, b, c, d, e, f\}$ with the relation

   $$\begin{aligned} \leq \ := \ & \{(a,a), (a,c), (a,d), (a,e), (a,f), (b,a), (b,b), (b,c), \\ & (b,d), (b,e), (b,f), (c,c), (c,e), (c,f), (d,d), (d,e), \\ & (d,f), (e,e), (f,f)\}. \end{aligned}$$

   (b) Let L and U be disjoint ordered sets. Construct $(P, \leq)$ as follows. $P := L \cup U$ and $a \leq b$ iff $a \in L$ and $b \in U$ or $a, b \in L$ and $a \leq_L b$ or $a, b \in U$ and $a \leq_U b$.

   (c) For a finite set X, consider the power set $\mathcal{P}(X)$ with the following relation. The set A is said to be **dominated** by the set B iff there is a k such that $|\{1, \ldots, k\} \cap A| < |\{1, \ldots, k\} \cap B|$ and for all $l < k$ we have $|\{1, \ldots, l\} \cap A| = |\{1, \ldots, l\} \cap B|$. Define $A \leq B$ iff $A = B$ or A is dominated by B.

   (d) Any tree $T = (V, E)$ ordered as follows. $x \leq y$ iff $x = y$ or there is a path from x to y that does not go through the root and the distance of x to the root is less than the distance of y to the root.

2. Let P be a set and let $\{\leq_i\}_{i \in I}$ be a family of order relations on P. Prove that $\bigcap_{i \in I} \leq_i$ is an order relation.

3. (Alternative definition of order.) Prove the following.

   (a) If $\leq$ is an order relation on P, then $<$ is an antireflexive (that is for all $p \in P$ we have $p \not< p$) and transitive relation.

(b) If $<$ is antireflexive and transitive, then $\leq := < \cup =$ is an order relation.

4. Prove that if $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1$ in an ordered set P, then $a_1 = a_2 = \cdots = a_n$.

5. Define the transitive closure of a relation on an arbitrary set. Then prove the analogue of Proposition 1.2.6.

6. Give an example of an infinite ordered set for which Proposition 1.2.7 fails. (Use the definition of the transitive closure from Exercise 5.)

7. Let $\prec$ be a relation that is contained in the transitive relation $\leq$. Prove that the transitive closure $\prec^t$ of $\prec$ is contained in $\leq$. Conclude that the transitive closure of a relation $\prec$ is the intersection of all transitive relations that contain $\prec$.

8. Draw the diagram of the ordered set.

   (a) The ordered set in Exercise 1a.

   (b) An ordered set as constructed in Exercise 1b with L and U isomorphic to ordered set as in Exercise 1a.

   (c) The set $\mathcal{P}(\{1, 2, 3, 4\})$ ordered as indicated in Exercise 1c.

9. Does the "spider" in Figure 1.1, part f) have a three-element subset C such that any two elements of C are comparable?

10. Prove that each of the following maps is order-preserving.

    (a) With the natural numbers $\mathbb{N}$ ordered as described in Example 1.1.2, part 6, define $f : \mathbb{N} \to \mathbb{N}$ by $f(x) := \begin{cases} \frac{x}{2}; & \text{if } x \text{ is even,} \\ \text{x}; & \text{if } x \text{ is odd.} \end{cases}$.

    (b) For the ordered set $P2$ in Figure 1.1, part b) define $F : P2 \to P2$ by $F(a) = b$, $F(b) = a$, $F(c) = c$, $F(d) = e$, $F(e) = d$, $F(f) = g$, $F(g) = f$, $F(h) = h$, $F(i) = k$, $F(k) = i$.

    (c) In the power set $\mathcal{P}(X)$ of any topological space (X, τ), the map $f : \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))$ that maps A to its closure.

11. Let P and Q be ordered sets.

    (a) Prove that *if P, Q are finite* the following are equivalent for a map $f : P \to Q$.

       i. f is order-preserving.

       ii. For all $a, b \in P$ we have that $a < b$ implies $f(a) \leq f(b)$.

       iii. For all $a, b \in P$ we have that $a \prec b$ implies $f(a) \leq f(b)$.

    (b) Prove that parts 11(a)i and 11(a)ii are equivalent for infinite ordered sets also.

(c) Find an example that shows that 11(a)iii is in general not equivalent to 11(a)i and 11(a)ii.

12. (The relation between covers and isomorphisms.)

 (a) Let $f : P \to Q$ be an isomorphism. Prove that $x \prec_P y$ implies $f(x) \prec_Q f(y)$.

 (b) Prove that if P and Q are finite, $f : P \to Q$ is an isomorphism iff f is bijective and $x \prec_P y$ is equivalent to $f(x) \prec_Q f(y)$.

 (c) Give an example of a bijective function $f : P \to Q$ such that $x \prec_P y$ is equivalent to $f(x) \prec_Q f(y)$ and yet f is not an isomorphism.

13. Let P be an ordered set and let $f : P \to P$ be an injective order-preserving self map.

 (a) Prove that if P is finite, then f is an automorphism.

 (b) Show that in general f need not be an automorphism.

14. Prove Proposition 1.3.6.

15. For two ordered sets $(P, \leq_P)$ and $(Q, \leq_Q)$ the ordered set $P \odot Q$ has the set $P \times Q$ as its points and $(p_1, q_1) \leq (p_2, q_2)$ iff $p_1 <_P p_2$ or $p_1 = p_2$ and $q_1 \leq_Q q_2$. Under what circumstances is $P \odot Q$ isomorphic to $Q \odot P$?

16. Let P be a finite ordered set and let $f : P \to P$ be order-preserving. Prove that if there is a $p \in P$ with $p \leq f(p)$, then f has a fixed point. Then find an infinite ordered set in which this result fails.

17. Prove that the ordered set $(\emptyset, \emptyset)$ does not have the fixed point property.

18. Prove that the following ordered sets have the fixed point property.

 (a) The ordered set in Figure 1.4 c)

 (b) The ordered set in Figure 1.1 b)

 (c) The ordered set in Figure 1.3

19. Prove that the range of the map in Figure 1.2 a) does not have the fixed point property.

20. Let P be an ordered set and let $\Psi : P \to P$ be an automorphism.

 (a) Prove that $f : P \to P$ has a fixed point iff $\Psi^{-1} \circ f \circ \Psi$ has a fixed point.

 (b) What general hypotheses can be imposed on Ψ to assure that f has a fixed point iff $\Psi \circ f \circ \Psi$ has a fixed point?

21. Let P, Q be ordered sets, let $f : P \to Q$ be order-preserving and injective and let P be finite. Prove that if $f[P]$ contains as many comparabilities as P, then f is an embedding.

22. It is clear that for every ordered set the identity and the constant function are order-preserving self maps. Call two order relations $\leq_1$ and $\leq_2$ on the same ground set P **perpendicular** iff the only order-preserving self maps they have in common are the constant functions and the identity. Prove that if $\leq_1$ and $\leq_2$ are perpendicular, then their intersection is $\leq_1 \cap \leq_2 = \{(x, x) : x \in P\}$.

23. (a) Find all nonisomorphic ordered sets with up to five elements. (Hint. The numbers of nonisomorphic sets are given in Remark 3 in Chapter 11.)
    (b) Verify that the reconstruction problem 1.5.9 is solvable for sets with four and five elements.
    (c) Attempt a positive solution of the reconstruction problem by hand or with a computer for small sets with more than five elements.[5]

24. An **isolated point** is a point in an ordered set that is only comparable to itself. Prove that ordered sets with an isolated point are reconstructible.

25. (a) Prove that the number of comparabilities in an ordered set is reconstructible,
    (b) For $p \in P$, the **degree** $\deg(p)$ is the number of elements that are comparable to p. Prove that for every card $P \setminus \{x\}$ of P the degree of the missing element x can be reconstructed.

26. Prove that the ordered set in Figure 1.1 c) is reconstructible.

Remarks and Open Problems

The "Remarks and Open Problems" sections are intended to give the reader some more background on the material that is discussed in the main body of the text. For this first section it consists solely of remarks, while in later sections we will list more and more open problems. To keep things organized and easy to refer to, these sections are normally enumerated lists.

1. The reader interested in the abstract underpinnings on objects and morphisms should look at category theory. A good introductory text is [3].

2. The fixed point property originated in topology using topological spaces and continuous functions (for a survey on the topological fixed point property, cf. [33]). It was apparently first studied for ordered sets by Tarski and

[5] A graduate student (MS) of the author's is at eleven at press time.

Davis (cf. [51, 260]) and has since steadily gained in attention. The essentials of Tarski's result are already present in joint work with Knaster mentioned in [145]. A complete characterization of the sets with the fixed point property might be beyond the realm of possibilities. Problems 1.4.3 and 1.4.8 have been proved to be (co-)NP-complete, cf. [62, 277], co-NP-completeness of problem 1.4.3 will be proved here in Theorem 12.4.5. Yet there are interesting connections to other fields (cf. [12, 192, 278], explored here in Chapters 6, 12 and Appendix B). Moreover it is possible to produce nice insights into the fixed point property for certain classes of sets (cf. [2, 51, 192, 216, 235, 241, 260]), which in turn can be used in applications (cf. [113, 114]) or to provide deeper structural insights into the theory of ordered sets (cf. [159, 162, 163] or Theorem 4.5.1 here).

3. The graph-theoretical analogue of an ordered subset is not the concept of a subgraph, but that of an *induced* subgraph. Subgraphs in graph theory are obtained by removing some vertices and possibly also some edges from the original graph. This works well, since graph theory has no a priori assumptions on the edges. In order theory, removal of comparabilities is not easy, since it might affect transitivity. Thus generating a substructure by removing comparabilities is not a widely used notion.

4. The reconstruction problem has its origins in graph theory and was recently also posed for ordered sets (cf. [127, 150, 151, 240]). According to "reliable sources" in [23] the problem is originally due to P.J. Kelly and S. Ulam who discovered the problem in 1942. The visualization via pictures on cards was suggested by Harary in 1964. For more background on the graph-theoretical reconstruction problem the reader may check for example the survey article [23] by Bondy and Hemminger. For a thorough survey of order reconstruction, cf. [213].

5. Note that in this section we have frequently used a certain luxury theorists have. In Definitions 1.2.4 and 1.2.5 we defined the diagram of an ordered set and the transitive closure of a relation. We have then shown how the diagram and the original order relation are linked via the taking of subsets and the formation of the transitive closure. In this fashion we are able to use these tools interchangeably and build the theory with them. The question we did not address was how to specifically translate between diagrams and order relations. What step-by-step procedure can one follow to translate one to the other and back? Similarly whenever in a proof we say "one can find", we do not address the issue how to find the object we are looking for. These issues, together with the algorithmic ramifications of the fixed point property and order isomorphism, are discussed in Chapter 12. Until then, we will freely use the mathematician's prerogative, which is that if an object exists or a transformation can be made, we will assume we can pick the object or make the transformation as necessary without worrying how long it might take us to find or execute it.