Bessel Functions

Bernd Schröder

Why are Bessel Functions Important?

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1. Parametric Bessel equations

$$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0$$

arise when the equations $\Delta u = k\frac{\partial u}{\partial t}$ and $\Delta u = k\frac{\partial^2 u}{\partial t^2}$ are solved with separation of variables in polar or cylindrical coordinates. The function $y(r)$ describes the radial part of the solution.

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2. Because 0 is a regular singular point of the equation, it is natural to attempt a solution using the method of Frobenius.

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$$x^2\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$$

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\end{aligned}$$

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x^2y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y &= 0 \\
x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} + \left(\lambda^2 x^2 - \nu^2\right) \sum_{n=0}^{\infty} c_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} \lambda^2 c_n x^{n+r+2}
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$$

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+\sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k + (k+r)c_k + \lambda^2c_{k-2} - \nu^2c_k\right]x^{k+r} &= 0
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\left(r(r-1)c_0+rc_0-\nu^2c_0\right)x^r+\left((r+1)rc_1+(r+1)c_1-\nu^2c_1\right)x^{r+1} &\\
+\sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k+(k+r)c_k+\lambda^2c_{k-2}-\nu^2c_k\right]x^{k+r} &= 0\\
\left(r^2-\nu^2\right)c_0x^r+\left((r+1)^2-\nu^2\right)c_1x^{r+1}+\sum_{k=2}^{\infty}\left[\left((k+r)^2-\nu^2\right)c_k+\lambda^2c_{k-2}\right]x^{k+r} &= 0
\end{aligned}$$

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Recurrence relation for $k \geq 2$:

$$\left((k+\nu)^2 - \nu^2\right)c_k + \lambda^2 c_{k-2} \quad = \quad 0$$

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Even Numbered Terms

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$$c_{2n} = -\frac{\lambda^2}{2n(2n+2\nu)}c_{2n-2}$$

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\begin{aligned}
c_{2n} &= -\frac{\lambda^2}{2n(2n+2\nu)}c_{2n-2} \\
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&= (-1)^3\frac{\left(\lambda^2\right)^3}{4^3 n(n-1)(n-2)(n+\nu)(n-1+\nu)(n-2+\nu)}c_{2(n-3)}
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&= (-1)^3\frac{\left(\lambda^2\right)^3}{4^3n(n-1)(n-2)(n+\nu)(n-1+\nu)(n-2+\nu)}c_{2(n-3)} \\
&\vdots \\
&= (-1)^n\frac{\left(\lambda^2\right)^n}{4^nn(n-1)\cdots 2\cdot 1\cdot(n+\nu)(n-1+\nu)\cdots(2+\nu)(1+\nu)}c_0
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&= (-1)^n\frac{\left(\lambda^2\right)^n}{4^nn(n-1)\cdots 2\cdot 1\cdot(n+\nu)(n-1+\nu)\cdots(2+\nu)(1+\nu)}c_0 \\
&= (-1)^n\frac{\Gamma(\nu+1)\left(\lambda^2\right)^n}{4^nn!\Gamma(n+\nu+1)}c_0
\end{aligned}$$

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&= \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n+\nu}}{2^{2n+\nu} n!\Gamma(n+\nu+1)} x^{2n+\nu}
\end{aligned}
$$

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$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu+1)\left(\lambda^2\right)^n}{4^n n!\Gamma(n+\nu+1)} x^{2n+\nu} \frac{\lambda^\nu}{2^\nu \Gamma(\nu+1)} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n+\nu}}{2^{2n+\nu} n!\Gamma(n+\nu+1)} x^{2n+\nu} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+\nu+1)} \left(\lambda \frac{x}{2}\right)^{2n+\nu}
\end{aligned}
$$

Bessel Functions of the First Kind

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$$J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

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$$
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J_\nu(x) &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} \\
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Both series are guaranteed to converge at least on $(0,\infty)$.

Bessel Functions of the First Kind

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For $\lambda > 0$ and $\nu > 0$ such that ν is not an integer, the general solution of the parametric Bessel equation

$$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0$$

is

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Of course, we are most interested in the solutions when ν is an integer.

Bessel Functions of the Second Kind

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For ν not an integer we define

$$Y_\nu(x) := \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

For $\nu = m$ an integer we define

$$Y_m(x) := \lim_{\nu\to m} Y_\nu(x).$$

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The functions Y_ν are called the **Bessel functions of the second kind.** For $\lambda > 0$ and *any* $\nu > 0$ the general solution of the parametric Bessel equation

$$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0$$

is

$$y(x) = c_1J_\nu(\lambda x) + c_2Y_\nu(\lambda x).$$

Vibrating Drum Membranes (Outline)

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1. The shape of a vibrating drum membrane can be modeled with the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=k\frac{\partial^2 u}{\partial t^2}$ and the condition that u is zero on the boundary.

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2. u is the displacement from equilibrium of the particle at (x, y) at time t.
3. The derivation is similar to that of the equation of an oscillating string. (Challenging exercise.)
4. For a round membrane, we solve the equation with separation of variables in polar coordinates, which leads to the Bessel equation and Bessel functions.

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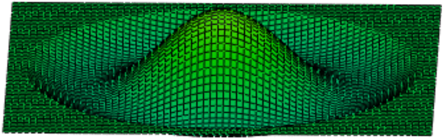

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