Cauchy-Euler Equations

Bernd Schröder

Definition and Solution Method

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$$y(x)=c_1x^1+c_2x^{-\frac{1}{2}}=c_1x+c_2\frac{1}{\sqrt{x}}$$

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y(x) &= \frac{5}{3}x-\frac{2}{3}\frac{1}{\sqrt{x}}
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Does $y(x) = \frac{5}{3}x - \frac{2}{3}\frac{1}{\sqrt{x}}$ Really Solve the Initial Value Problem

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If $g(x) = 0$, then the equation is called **homogeneous**.
3. The substitution $t = \ln(x)$ turns the Cauchy-Euler equation
$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$
for $x > 0$ into a linear differential equation with constant coefficients.

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5. If $r_k = a_k + ib_k$ is complex then $y(x) = x^{a_k} \cos\left(b_k \ln(x)\right)$ and $y(x) = x^{a_k} \sin\left(b_k \ln(x)\right)$ solve the differential equation.

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6. If $(r - r_k)^j$ is a factor of $p(r)$, then x^{r_k}, $\ln(x) x^{r_k}$, $\ldots$, $\big(\ln(x)\big)^{j-1} x^{r_k}$ solve the differential equation. (If r_k is complex we need to multiply the solutions from 5 with the power of the logarithm.)

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7. The general solution is a linear combination of the solutions above with generic coefficients $c_1, \ldots, c_n$.