Runge Kutta Methods

Bernd Schröder

Errors in Euler's Method

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1. **Taylor's Formula**. If the function y is $n+1$ times differentiable, then for any h there is a c between x and $x+h$ so that

$$y(x+h)=y(x)+y'(x)h+\frac{y''(x)}{2!}h^2+\cdots+\frac{y^{(n)}(x)}{n!}h^n+\frac{y^{(n+1)}(c)}{(n+1)!}h^{n+1}.$$

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4. So the error in each step is proportional to $(\Delta x)^2$.
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4. If we can capture more than the first two terms of the Taylor expansion, we could get a global error proportional to $(\Delta x)^n$. This would be good, because Δx is usually small, so a higher power of Δx would be even smaller.

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&= y(x)+\left(\frac{1}{2}y'(x)+\frac{1}{2}y'(x+h)\right)h+\left(\frac{y'''(c)}{3!}-\frac{y'''(\tilde{c})}{4}\right)h^3
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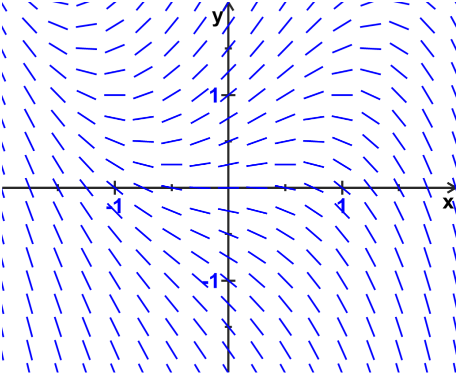

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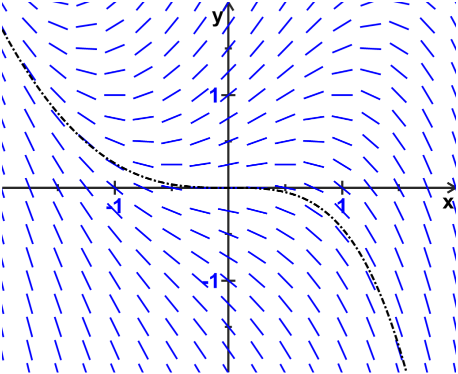

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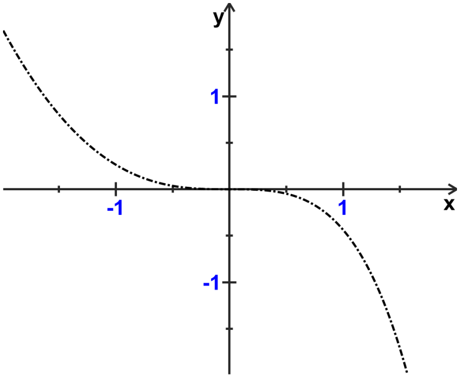

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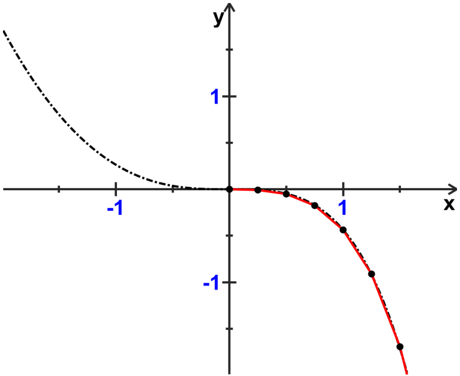

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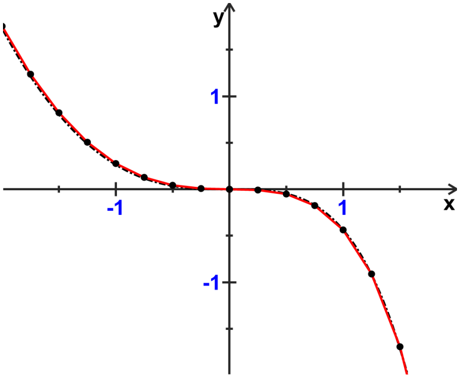

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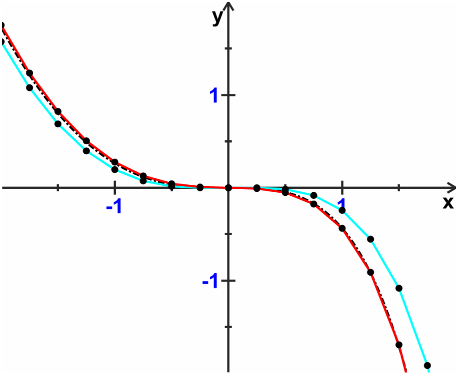

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1. The increment in the improved Euler method $\frac{1}{2}F\big(x,y(x)\big)\Delta x+\frac{1}{2}F\big(x+\Delta x,y_{\mathrm{Euler}}(x+\Delta x)\big)\Delta x$ looks like the increment in the trapezoidal rule in numerical integration: $\frac{1}{2}\big(f(x_i)+f(x_{i+1})\big)\Delta x$

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4. Match as many terms of Taylor's formula as possible. The remainder term gives the order of the error.
5. The global error in Simpson's rule is $\sim(\Delta x)^4$, so we must match the first four terms of the Taylor expansion.

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k_3 &= F\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_2\right)\Delta x, \\
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\end{aligned}$$

y_n approximates the value of the solution y at x_n. One step error $\sim (\Delta x)^5$, global error $\sim (\Delta x)^4$.

The most common Runge-Kutta Method

Step length Δx, initial values (x_0, y_0),

$$\begin{aligned} x_{n+1} &:= x_n + \Delta x \\ y_{n+1} &= y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \qquad \text{where} \\ k_1 &= F(x_n, y_n)\Delta x, \\ k_2 &= F\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1\right)\Delta x, \\ k_3 &= F\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_2\right)\Delta x, \\ k_4 &= F\left(x_n + \Delta x, y_n + k_3\right)\Delta x, \end{aligned}$$

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Fourth Order Runge-Kutta Method

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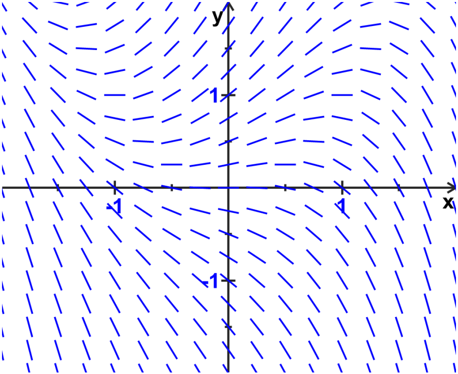
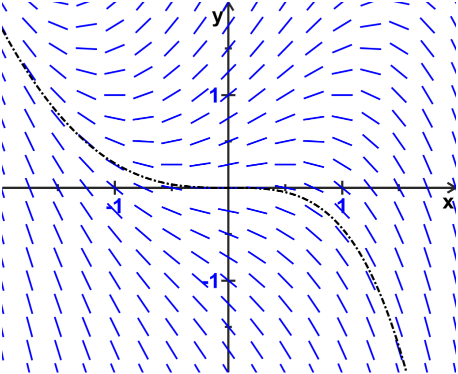


$$y' = y - x^2, \; y(0) = 0$$

Fourth Order Runge-Kutta Method



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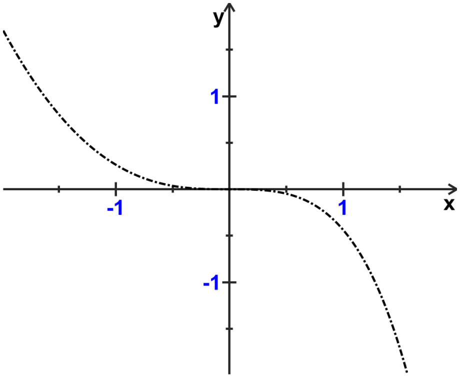

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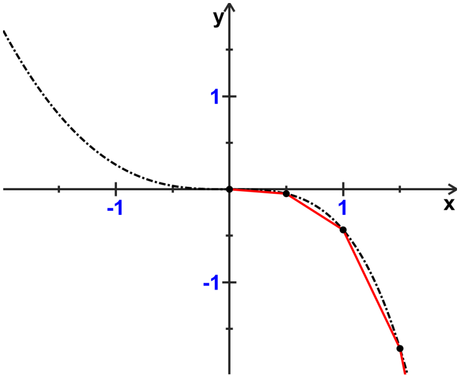

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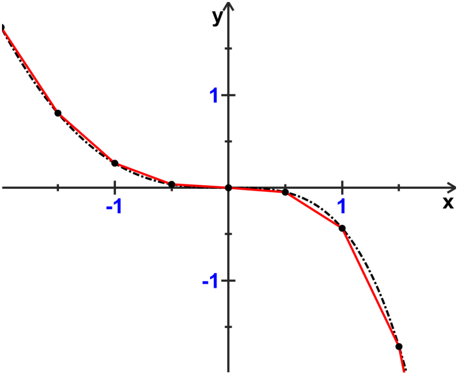

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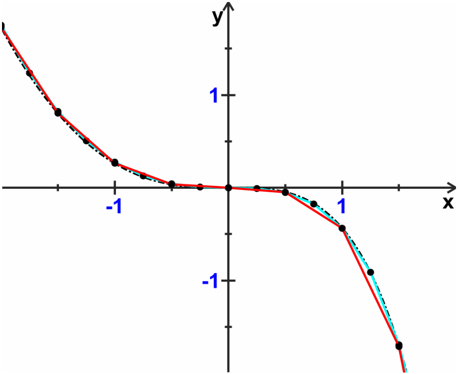

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Reminder for Spreadsheet Implementation

Step length: Δx. Initial values: (x_0, y_0).

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A j^{th} order **Runge-Kutta procedure** computes an approximate solution to the differential equation $y' = F(x,y)$ by computing a sequence of values $y_{n+1} := y_n + a_1k_1 + a_2k_2 + \cdots + a_mk_m$ so that y_{n+1} agrees with the j^{th} order Taylor polynomial of y at the previous evaluation point.

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Setting up a formula for increments to match a Taylor polynomial actually leads to a system of equations for the parameters in the setup of the approximation formulas. So there is more than one second order Runge-Kutta method. Same goes for higher orders. Some of this freedom can be used to improve performance for certain types of equations.