Runge Kutta Methods

Bernd Schröder
Errors in Euler’s Method

1. Taylor’s Formula
   If the function $y$ is $n+1$ times differentiable, then for any $h$ there is a $c$ between $x$ and $x+h$ so that
   \[
   y(x+h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2 + \cdots + \frac{y^{(n)}(x)}{n!}h^n + \frac{y^{(n+1)}(c)}{(n+1)!}h^{n+1}.
   \]

2. Euler’s method
   For $y' = F(x, y)$, $y(x) = y_0$ we use that
   \[
   y(x+\Delta x) \approx y(x) + y'(x)\Delta x = y_0 + F(x, y_0)\Delta x.
   \]

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y(x + \Delta x) \approx y(x) + y'(x)\Delta x
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= y_0 + F(x, y_0)\Delta x
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\[
= y_{\text{Euler}}(x + \Delta x)
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Errors in Euler’s Method

But we know that
\[ y(x + \Delta x) = y(x) + y'(x) \Delta x + y''(c) \frac{\Delta x^2}{2} = y_{Euler}(x + \Delta x) + y''(c) \frac{\Delta x^2}{2} \]

So the error in each step is proportional to \( (\Delta x)^2 \).

Summing the errors for \( b - a \) steps gives an overall error proportional to \( \Delta x \).

(Details are more subtle than it looks.)
Errors in Euler’s Method

3. But we know that

\[ y(x + \Delta x) = y(x) + y'(x) \Delta x + \frac{y''(c)}{2!} (\Delta x)^2 = y_{\text{Euler}}(x + \Delta x) + \frac{y''(c)}{2!} (\Delta x)^2 \]

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How can we Shrink the Error?

1. Shrinking $\Delta x$ is costly.
2. So a formula with a smaller error would be nice.
3. The global error's proportionality to $\Delta x$ in Euler's method came from the fact that Euler's method uses the first two terms of the Taylor expansion.
4. If we can capture more than the first two terms of the Taylor expansion, we could get a global error proportional to $(\Delta x)^n$. This would be good, because $\Delta x$ is usually small, so a higher power of $\Delta x$ would be even smaller.
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Improving Euler’s Method

We'll get $y''$ from the expansion for $y'$.

\[ y'(x+h) = y'(x) + y''(x)h + y'''(\bar{c}) \frac{h^2}{2!} \]

\[ y''(x) = y'(x+h) - y'(x)h - y'''(\bar{c}) \frac{h^2}{2!} \]

\[ y(x+h) = y(x) + y'(x)h + \frac{1}{2} y''(x)h^2 + \frac{1}{6} y'''(c) h^3 \]
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\]

\[
= y(x) + y'(x)h + \frac{1}{2}h^2 \left( \frac{y'(x + h) - y'(x)}{h} - \frac{y'''(\tilde{c})}{2!}h \right) + \frac{y'''(c)}{3!}h^3
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\]

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= y(x) + y'(x)h + \frac{1}{2} y'(x+h)h - \frac{1}{2} y'(x)h - \frac{1}{2} \frac{y'''(\tilde{c})}{2!} h^3 + \frac{y'''(c)}{3!} h^3
\]

\[
= y(x) + \left( \frac{1}{2} y'(x) + \frac{1}{2} y'(x+h) \right) h + \left( \frac{y'''(c)}{3!} - \frac{y'''(\tilde{c})}{4} \right) h^3
\]
Error Analysis

Improved Euler Method

Improved Euler Method

For a given step length $\Delta x$, with initial values $(x_0, y_0)$, we compute the two recursively defined sequences 

\[
\begin{align*}
x_n & = x_{n-1} + \Delta x \\
y_n & = y_{n-1} + \frac{1}{2} k_1 + \frac{1}{2} k_2,
\end{align*}
\]

where 

\[
k_1 = F(x_n, y_n),
\]

\[
k_2 = F(x_n + \Delta x, y_n + k_1).
\]

The value $y_n$ will be an approximation for the value of the solution $y$ at $x_n$.

(Formulated to make the transition to Runge-Kutta methods easier.)

The one step error is proportional to $(\Delta x)^3$ and the global error is proportional to $(\Delta x)^2$.

(Again we omit the considerable details.)

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Runge Kutta Methods
Improved Euler Method

For a given step length $\Delta x$, with initial values $(x_0, y_0)$, we compute the two recursively defined sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ via

$$x_{n+1} = x_n + \Delta x,$$
$$y_{n+1} = y_n + \frac{1}{2}k_1 + \frac{1}{2}k_2,$$

where

$$k_1 = F(x_n, y_n) \Delta x,$$
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For a given step length $\Delta x$, with initial values $(x_0, y_0)$, we compute the two recursively defined sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ via

$$x_{n+1} := x_n + \Delta x$$
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The value $y_n$ will be an approximation for the value of the solution $y$ at $x_n$. (Formulated to make the transition to Runge-Kutta methods easier.) The one step error is proportional to $(\Delta x)^3$ and the global error is proportional to $(\Delta x)^2$. 
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Improved Euler Method versus Euler’s Method
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\[ y' = y - x^2, \quad y(0) = 0 \]
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Reminder for Spreadsheet Implementation
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Step length: $\Delta x$. Initial values: $(x_0, y_0)$.

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k_1 = F(x_n, y_n) \Delta x,
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The value $y_n$ will be an approximation for the value of the solution $y$ at $x_n$. 
Further Improvements

1. The increment in the improved Euler method looks like the increment in the trapezoidal rule in numerical integration:
   \[ F(x, y(x)) \Delta x + \frac{1}{2} F(x + \Delta x, y_{\text{Euler}}(x + \Delta x)) \Delta x \]

2. Simpson's rule is more accurate than the trapezoidal rule. Its increment is
   \[ f(x_i) \Delta x + \frac{4}{6} f(x_{i+1/2}) \Delta x + f(x_{i+1}) \Delta x \]

3. So how do we translate this into a formula for differential equations that has high accuracy?

4. Match as many terms of Taylor's formula as possible. The remainder term gives the order of the error.

5. The global error in Simpson's rule is \( \sim (\Delta x)^4 \), so we must match the first four terms of the Taylor expansion.
Further Improvements

1. The increment in the improved Euler method
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   \frac{1}{2}F(x, y(x)) \Delta x + \frac{1}{2}F(x + \Delta x, y_{\text{Euler}}(x + \Delta x)) \Delta x
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\[ f(x_i) \frac{\Delta x}{6} + 4f \left(x_{i+\frac{1}{2}}\right) \frac{\Delta x}{6} + f(x_{i+1}) \frac{\Delta x}{6}. \]
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3. So how do we translate this into a formula for differential equations that has high accuracy?

4. Match as many terms of Taylor’s formula as possible. The remainder term gives the order of the error.

5. The global error in Simpson’s rule is \( \sim (\Delta x)^4 \), so we must match the first four terms of the Taylor expansion.
The most common Runge-Kutta Method

\[ \begin{align*}
\text{Step length} & \quad \Delta x, \quad \text{initial values} \quad (x_0, y_0), \\
\text{at} \quad x_{n+1} & = x_n + \Delta x, \\
y_{n+1} & \approx y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4),
\end{align*} \]

where
\[ k_1 = F(x_n, y_n) \Delta x, \]
\[ k_2 = F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1) \Delta x, \]
\[ k_3 = F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2) \Delta x, \]
\[ k_4 = F(x_n + \Delta x, y_n + k_3) \Delta x. \]

One step error \( \sim (\Delta x)^5 \), global error \( \sim (\Delta x)^4 \).

(Considerable details omitted.)
The most common **Runge-Kutta Method**

Step length $\Delta x$, initial values $(x_0, y_0)$,

\[ y_{n+1} \approx y_n + \frac{\Delta x}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]

where

\[ k_1 = F(x_n, y_n) \Delta x, \]

\[ k_2 = F(x_n + \frac{\Delta x}{2}, y_n + \frac{k_1}{2}) \Delta x, \]

\[ k_3 = F(x_n + \frac{\Delta x}{2}, y_n + \frac{k_2}{2}) \Delta x, \]

\[ k_4 = F(x_n + \Delta x, y_n + k_3) \Delta x. \]
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

$$x_{n+1} := x_n + \Delta x$$
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

\[
\begin{align*}
    x_{n+1} & := x_n + \Delta x \\
    y_{n+1} & = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4),
\end{align*}
\]

where

\[
\begin{align*}
    k_1 &= F(x_n, y_n) \\
    k_2 &= F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1) \\
    k_3 &= F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2) \\
    k_4 &= F(x_n + \Delta x, y_n + k_3)
\end{align*}
\]
The most common Runge-Kutta Method

Step length \( \Delta x \), initial values \((x_0, y_0)\),

\[
x_{n+1} := x_n + \Delta x
\]

\[
y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad \text{where}
\]

\[
k_1 = F(x_n, y_n) \Delta x,
\]

\[
k_2 = F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1) \Delta x,
\]

\[
k_3 = F(x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2) \Delta x
\]

\[
k_4 = F(x_n + \Delta x, y_n + k_3) \Delta x,
\]
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

$$x_{n+1} := x_n + \Delta x$$

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad \text{where}$$

$$k_1 = F(x_n, y_n)\Delta x,$$

$$k_2 = F\left(x_n + \frac{1}{2}\Delta x, y_n + \frac{1}{2}k_1\right) \Delta x,$$
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

\[
\begin{align*}
x_{n+1} & := x_n + \Delta x \\
y_{n+1} & = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad \text{where} \\
k_1 & = F(x_n, y_n) \Delta x, \\
k_2 & = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1 \right) \Delta x, \\
k_3 & = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2 \right) \Delta x,
\end{align*}
\]
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Step length $\Delta x$, initial values $(x_0, y_0)$,

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\begin{align*}
    x_{n+1} & := x_n + \Delta x \\
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    k_4 & = F \left( x_n + \Delta x, y_n + k_3 \right) \Delta x,
\end{align*}
\]

One step error $\sim (\Delta x)^5$, global error $\sim (\Delta x)^4$. (Considerable details omitted.)
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

\[ x_{n+1} := x_n + \Delta x \]
\[ y_{n+1} = y_n + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right), \quad \text{where} \]
\[ k_1 = F(x_n, y_n) \Delta x, \]
\[ k_2 = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1 \right) \Delta x, \]
\[ k_3 = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2 \right) \Delta x, \]
\[ k_4 = F \left( x_n + \Delta x, y_n + k_3 \right) \Delta x, \]

$y_n$ approximates the value of the solution $y$ at $x_n$. 
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

$$x_{n+1} := x_n + \Delta x$$

$$y_{n+1} = y_n + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right),$$

where

$$k_1 = F(x_n, y_n) \Delta x,$$

$$k_2 = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1 \right) \Delta x,$$

$$k_3 = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2 \right) \Delta x,$$

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$y_n$ approximates the value of the solution $y$ at $x_n$. One step error $\sim (\Delta x)^5$, global error $\sim (\Delta x)^4$.
The most common Runge-Kutta Method

Step length $\Delta x$, initial values $(x_0, y_0)$,

$$x_{n+1} := x_n + \Delta x$$

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$$k_4 = F \left( x_n + \Delta x, y_n + k_3 \right) \Delta x,$$

$y_n$ approximates the value of the solution $y$ at $x_n$. One step error $\sim (\Delta x)^5$, global error $\sim (\Delta x)^4$. (Considerable details omitted.)
Fourth Order Runge-Kutta Method
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\[ y' = y - x^2, \quad y(0) = 0 \]
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Fourth Order Runge-Kutta Method

\[ y' = y - x^2, \quad y(0) = 0 \]
Reminder for Spreadsheet Implementation

Step length: $\Delta x$. Initial values: $(x_0, y_0)$.

\[
\begin{align*}
  x_{n+1} & := x_n + \Delta x \\
  y_{n+1} & = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4), \quad \text{where} \\
  k_1 & = F(x_n, y_n) \Delta x, \\
  k_2 & = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_1 \right) \Delta x, \\
  k_3 & = F \left( x_n + \frac{1}{2} \Delta x, y_n + \frac{1}{2} k_2 \right) \Delta x, \\
  k_4 & = F \left( x_n + \Delta x, y_n + k_3 \right) \Delta x,
\end{align*}
\]

$y_n$ will be an approximation for the value of the solution $y$ at $x_n$. 
General Runge-Kutta Methods

A \( j \)th order Runge-Kutta procedure computes an approximate solution to the differential equation

\[ y' = F(x, y) \]

by computing a sequence of values

\[ y_{n+1} = y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m \]

so that \( y_{n+1} \) agrees with the \( j \)th order Taylor polynomial of \( y \) at the previous evaluation point.

The Euler method is a first order Runge-Kutta procedure. Improved Euler method: second order Runge-Kutta. Have seen a fourth order Runge-Kutta procedure.

Setting up a formula for increments to match a Taylor polynomial actually leads to a system of equations for the parameters in the setup of the approximation formulas. So there is more than one second order Runge-Kutta method. Same goes for higher orders. Some of this freedom can be used to improve performance for certain types of equations.
General Runge-Kutta Methods

A $j^{\text{th}}$ order Runge-Kutta procedure computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{\text{th}}$ order Taylor polynomial of $y$ at the previous evaluation point.
General Runge-Kutta Methods

A \( j \)th order **Runge-Kutta procedure** computes an approximate solution to the differential equation \( y' = F(x, y) \) by computing a sequence of values

\[
y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m
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General Runge-Kutta Methods

A $j^{\text{th}}$ order **Runge-Kutta procedure** computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{\text{th}}$ order Taylor polynomial of $y$ at the previous evaluation point.

The Euler method is a first order Runge-Kutta procedure. Improved Euler method: second order Runge-Kutta.
General Runge-Kutta Methods

A $j^{th}$ order Runge-Kutta procedure computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{th}$ order Taylor polynomial of $y$ at the previous evaluation point.

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General Runge-Kutta Methods

A $j^{th}$ order **Runge-Kutta procedure** computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1k_1 + a_2k_2 + \cdots + a_mk_m$ so that $y_{n+1}$ agrees with the $j^{th}$ order Taylor polynomial of $y$ at the previous evaluation point.

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Setting up a formula for increments to match a Taylor polynomial actually leads to a system of equations for the parameters in the setup of the approximation formulas.
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A $j^{th}$ order Runge-Kutta procedure computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{th}$ order Taylor polynomial of $y$ at the previous evaluation point.

The Euler method is a first order Runge-Kutta procedure.

Improved Euler method: second order Runge-Kutta.

Have seen a fourth order Runge-Kutta procedure.

Setting up a formula for increments to match a Taylor polynomial actually leads to a system of equations for the parameters in the setup of the approximation formulas. So there is more than one second order Runge-Kutta method.
General Runge-Kutta Methods

A $j^{\text{th}}$ order **Runge-Kutta procedure** computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{\text{th}}$ order Taylor polynomial of $y$ at the previous evaluation point.

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General Runge-Kutta Methods

A $j^{\text{th}}$ order Runge-Kutta procedure computes an approximate solution to the differential equation $y' = F(x, y)$ by computing a sequence of values $y_{n+1} := y_n + a_1 k_1 + a_2 k_2 + \cdots + a_m k_m$ so that $y_{n+1}$ agrees with the $j^{\text{th}}$ order Taylor polynomial of $y$ at the previous evaluation point.

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Setting up a formula for increments to match a Taylor polynomial actually leads to a system of equations for the parameters in the setup of the approximation formulas. So there is more than one second order Runge-Kutta method. Same goes for higher orders. Some of this freedom can be used to improve performance for certain types of equations.