Method of Frobenius – A Problematic Case

Bernd Schröder

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2. But P and Q cannot be arbitrary: $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ must be analytic at x_0.
3. Instead of a series solution $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, we obtain a solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$.

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2. But P and Q cannot be arbitrary: $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ must be analytic at x_0.
3. Instead of a series solution $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, we obtain a solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$.
4. As for series solutions, we substitute the series and its derivatives into the equation to obtain an equation for r and a set of equations for the c_n.

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5. These equations will allow us to compute r and the c_n.
6. For each value of r (typically there are two), we can compute the solution just like for series.
7. The method of Frobenius is guaranteed to produce *one* solution. But when the two values for r differ by an integer, it may not produce two linearly independent solutions.

Frobenius Solution for $x^2y'' + xy' + \left(x^2 - 4\right)y = 0$

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+\sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k+(k+r)c_k+c_{k-2}-4c_k\right]x^{k+r} &= 0\\
\left(r^2-4\right)c_0x^r+\left((r+1)^2-4\right)c_1x^{r+1}+\sum_{k=2}^{\infty}\left[\left((k+r)^2-4\right)c_k+c_{k-2}\right]x^{k+r} &= 0
\end{aligned}$$

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Recurrence relation for $k \geq 2$:

$$\left((k+2)^2 - 4\right)c_k + c_{k-2} \quad = \quad 0$$

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Recurrence relation for $k \geq 2$:

$$\begin{aligned} \left((k+2)^2 - 4\right)c_k + c_{k-2} &= 0 \\ c_k = -\frac{c_{k-2}}{(k+2)^2 - 4} &= -\frac{c_{k-2}}{k(k+4)} \end{aligned}$$

Even Numbered Terms

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$$c_{2n} = -\frac{c_{2n-2}}{2n(2n+4)}$$

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$$\begin{aligned} c_{2n} &= -\frac{c_{2n-2}}{2n(2n+4)} \\ &= -\frac{c_{2(n-1)}}{4n(n+2)} \end{aligned}$$

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$$\begin{aligned} c_{2n} &= -\frac{c_{2n-2}}{2n(2n+4)} \\ &= -\frac{c_{2(n-1)}}{4n(n+2)} \\ &= (-1)^2 \frac{c_{2(n-2)}}{4^2 n(n-1)(n+2)(n+1)} \end{aligned}$$

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&= (-1)^3 \frac{c_{2(n-3)}}{4^3 n(n-1)(n-2)(n+2)(n+1)n}
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&= (-1)^n \frac{c_0}{4^n n(n-1)\cdots 2\cdot 1\cdot (n+2)(n+1)\cdots 4\cdot 3}
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&= (-1)^n\frac{c_0}{4^n n(n-1)\cdots 2\cdot 1\cdot (n+2)(n+1)\cdots 4\cdot 3} \\
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\end{aligned}
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$$
y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{2}{4^n n!(n+2)!} x^{2n+2}
$$

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c_k &= -\frac{c_{k-2}}{(k-2)^2-4} = -\frac{c_{k-2}}{k(k-4)}
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\begin{aligned}
c_0 &:= 1 \\
c_1\left((-2+1)^2-4\right)=0, &\qquad c_1=0 \\
\left((k-2)^2-4\right)c_k+c_{k-2} &= 0 \\
c_k &= -\frac{c_{k-2}}{(k-2)^2-4} = -\frac{c_{k-2}}{k(k-4)} \\
c_2 &= -\frac{c_0}{2(-2)}
\end{aligned}
$$

r=-2

$$
\begin{aligned}
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c_k &= -\frac{c_{k-2}}{(k-2)^2-4} = -\frac{c_{k-2}}{k(k-4)} \\
c_2 &= -\frac{c_0}{2(-2)} = \frac{1}{4}
\end{aligned}
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\end{aligned}
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c_2 &= -\frac{c_0}{2(-2)}=\frac{1}{4} \\
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c_3 &= -\frac{c_1}{3(-1)} = 0 \\
c_4 &= -\frac{c_2}{4}
\end{aligned}
$$

r=-2

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c_2 &= -\frac{c_0}{2(-2)} = \frac{1}{4} \\
c_3 &= -\frac{c_1}{3(-1)} = 0 \\
c_4 &= -\frac{c_2}{4\cdot 0} ???
\end{aligned}$$

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

$$x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 \stackrel{?}{=} 0$$

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$$x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 \stackrel{?}{=} 0$$

$$x^2\left(\frac{6}{x^4}\right) + x\left(-\frac{2}{x^3}\right) + \left(x^2 - 4\right)\left(\frac{1}{x^2} + \frac{1}{4}\right) \stackrel{?}{=} 0$$

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

$$
\begin{aligned}
x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 &\stackrel{?}{=} 0 \\
x^2\left(\frac{6}{x^4}\right) + x\left(-\frac{2}{x^3}\right) + \left(x^2 - 4\right)\left(\frac{1}{x^2} + \frac{1}{4}\right) &\stackrel{?}{=} 0 \\
\frac{6}{x^2} - \frac{2}{x^2} + 1 + \frac{1}{4}x^2 - \frac{4}{x^2} - 1 &\stackrel{?}{=} 0
\end{aligned}
$$

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

$$\begin{aligned}
x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 &\stackrel{?}{=} 0 \\
x^2\left(\frac{6}{x^4}\right) + x\left(-\frac{2}{x^3}\right) + \left(x^2 - 4\right)\left(\frac{1}{x^2} + \frac{1}{4}\right) &\stackrel{?}{=} 0 \\
\frac{6}{x^2} - \frac{2}{x^2} + 1 + \frac{1}{4}x^2 - \frac{4}{x^2} - 1 &\stackrel{?}{=} 0 \\
\frac{1}{4}x^2 &\stackrel{?}{=} 0
\end{aligned}$$

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

$$\begin{aligned}
x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 &\stackrel{?}{=} 0 \\
x^2 \left(\frac{6}{x^4}\right) + x\left(-\frac{2}{x^3}\right) + \left(x^2 - 4\right)\left(\frac{1}{x^2} + \frac{1}{4}\right) &\stackrel{?}{=} 0 \\
\frac{6}{x^2} - \frac{2}{x^2} + 1 + \frac{1}{4}x^2 - \frac{4}{x^2} - 1 &\stackrel{?}{=} 0 \\
\frac{1}{4}x^2 &\stackrel{?}{=} 0
\end{aligned}$$

NO!

Trying $y_2 = \frac{1}{x^2} + \frac{1}{4}$

$$
\begin{aligned}
x^2 y_2'' + x y_2' + \left(x^2 - 4\right) y_2 &\stackrel{?}{=} 0 \\
x^2 \left(\frac{6}{x^4}\right) + x\left(-\frac{2}{x^3}\right) + \left(x^2 - 4\right)\left(\frac{1}{x^2} + \frac{1}{4}\right) &\stackrel{?}{=} 0 \\
\frac{6}{x^2} - \frac{2}{x^2} + 1 + \frac{1}{4}x^2 - \frac{4}{x^2} - 1 &\stackrel{?}{=} 0 \\
\frac{1}{4}x^2 &\stackrel{?}{=} 0
\end{aligned}
$$

NO!

(Reduction of Order can help.)