# Laplace Transforms and Convolutions

Bernd Schröder

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| Original DE & IVP | $\xrightarrow{\mathscr{L}}$ | Algebraic equation for the Laplace transform |
|---|---|---|

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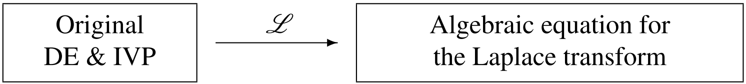
Time Domain ($t$) Transform domain ($s$)



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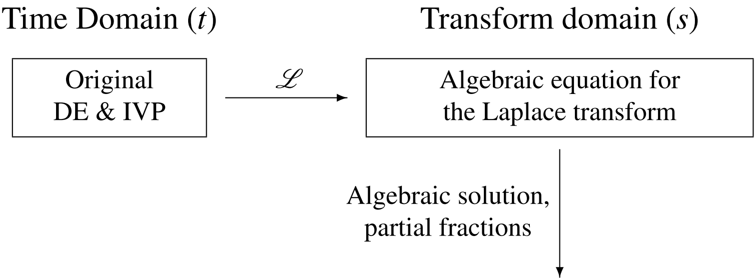

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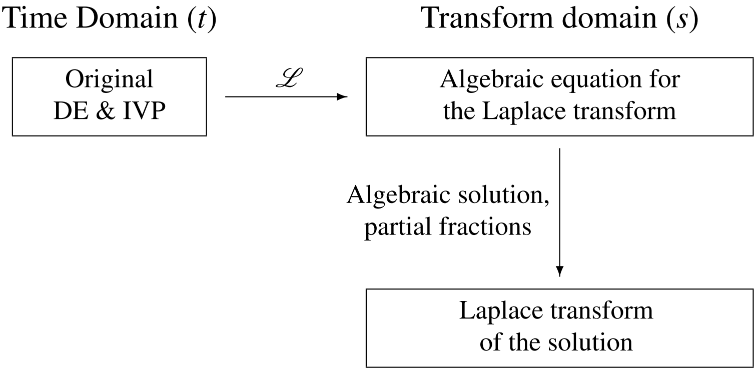

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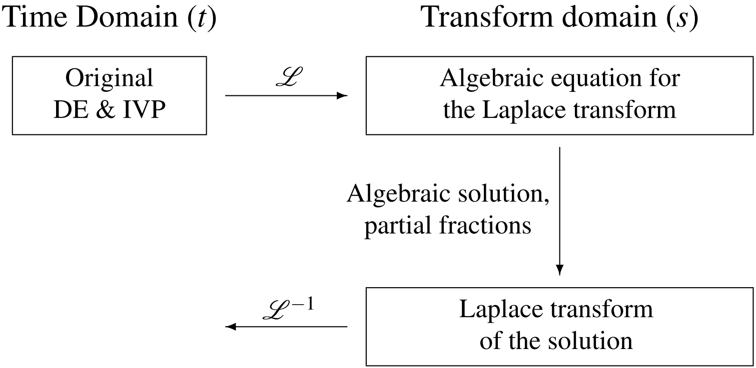

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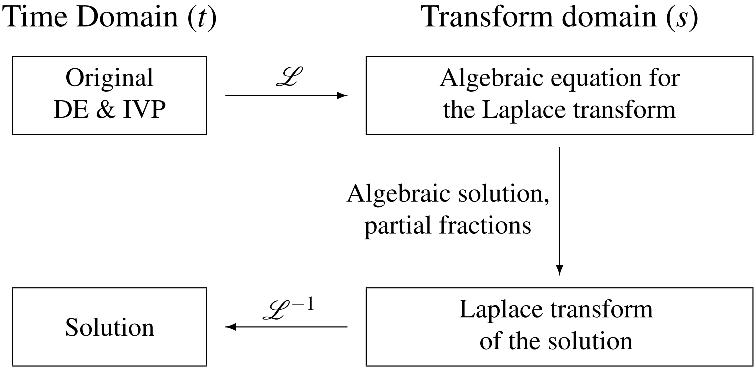

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3. The convolution of the functions $f(t)$ and $g(t)$ is

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3. The convolution of the functions $f(t)$ and $g(t)$ is $f*g(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$ and $\mathscr{L}(f*g)=FG$.
4. So it is possible to avoid transforming the forcing term, but the price we pay is that the solution is represented as an integral.

# The Convolution Can Be Useful When Larger Systems are Analyzed

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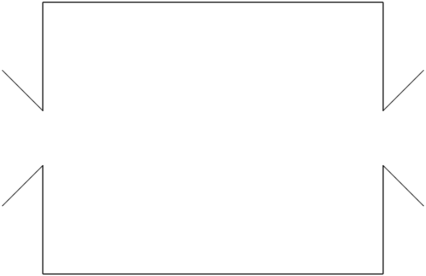

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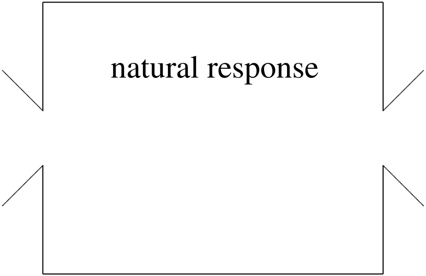

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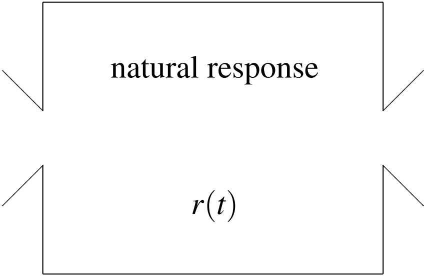

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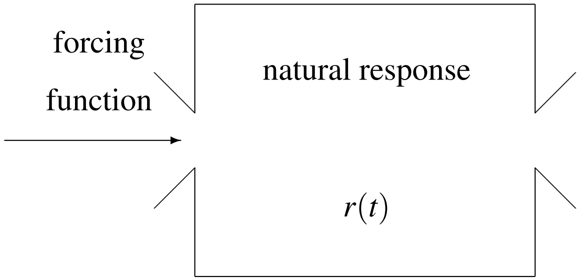

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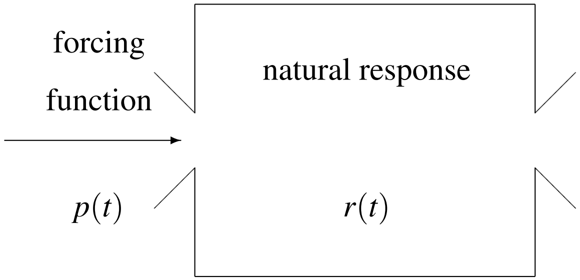

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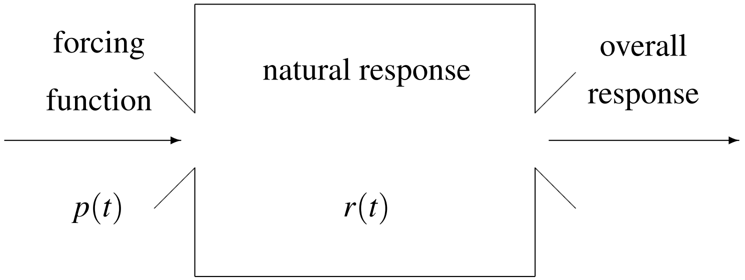

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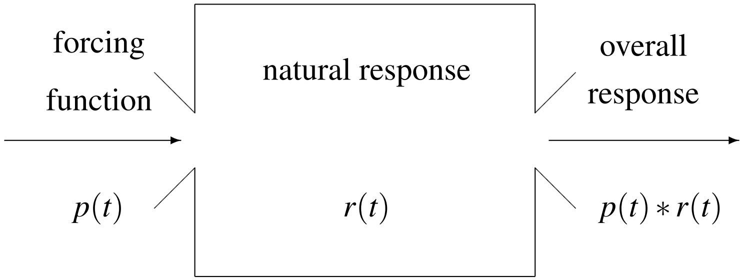

# Solve the Initial Value Problem

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y(t) &= p(t) * \frac{1}{3}e^{-\frac{2}{3}t}
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y(t) &= p(t)*\frac{1}{3}e^{-\frac{2}{3}t}=\frac{1}{3}\int_0^t \left|\sin(\tau)\right|e^{-\frac{2}{3}(t-\tau)}\,d\tau
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## Does $y = \frac{1}{3} \int_0^t \left| \sin(\tau) \right| e^{-\frac{2}{3}(t-\tau)} \, d\tau$ Really Solve the Initial Value Problem $3y' + 2y = \left| \sin(t) \right|, \; y(0) = 0$ ?

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Initial value

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$3y'+2y$

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&= \frac{1}{3}\left|\sin(t)\right| - \frac{2}{3}y \\
3y' + 2y &= 3\left(\frac{1}{3}\left|\sin(t)\right| - \frac{2}{3}y\right) + 2y
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3y' + 2y &= 3\left(\frac{1}{3}\left|\sin(t)\right| - \frac{2}{3}y\right) + 2y = \left|\sin(t)\right| \qquad \checkmark
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# Comparing Output to Input

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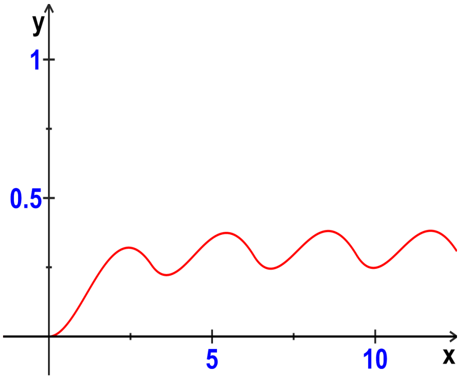

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