Introducing Laplace Transforms

Bernd Schröder

The Laplace Transform

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2. But simply integrating both sides of $ay'' + by' + cy = F(t)$ does not help.
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4. But that’s just not possible with indefinite integrals.
5. Some transform of the equation that has a similar effect would be nice.
6. Let $f(t)$ be a function on $[0, \infty)$. The function

$$F(t) := \mathscr{L}\{f\}(s) := \int_0^\infty f(t)e^{-st}\, dt$$

(if it exists) is called the **Laplace transform** of f.

The Laplace Transform “Undoes Derivatives”

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&= 0 - y(0) + s\int_0^{\infty} y(t)e^{-st}\,dt \\
&= s\mathscr{L}\left\{y\right\}(s) - y(0)
\end{aligned}$$

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$$\mathscr{L}\left\{y''\right\}(s) \;=\; \int_0^\infty y''(t)e^{-st}\,dt$$

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The Laplace Transform "Acts Like an Integral"

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\mathcal{L}\{af+bg\}(s) &= \int_0^\infty (af+bg)(t)e^{-st}\,dt \\
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&= a\int_0^\infty f(t)e^{-st}\,dt
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&= \int_0^\infty af(t)e^{-st}\,dt+\int_0^\infty bg(t)e^{-st}\,dt \\
&= a\int_0^\infty f(t)e^{-st}\,dt+b\int_0^\infty g(t)e^{-st}\,dt
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&= a\mathscr{L}\{f\}(s)+b\mathscr{L}\{g\}(s)
\end{aligned}$$

Laplace Transforms Turn Initial Value Problems into Algebra Problems in the s-Domain

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Time Domain (t)

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Time Domain (t)

Original DE & IVP

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Time Domain (t)

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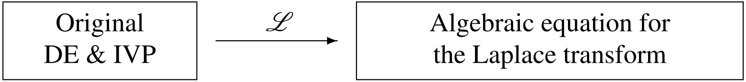
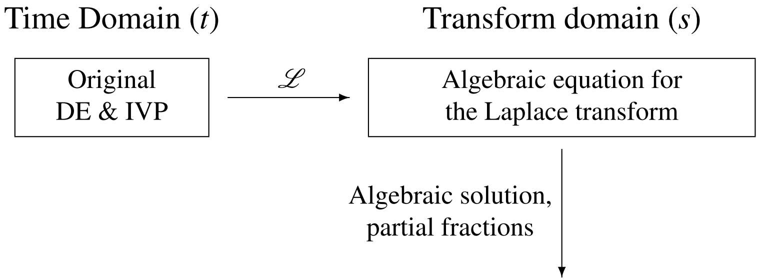
Time Domain (t)



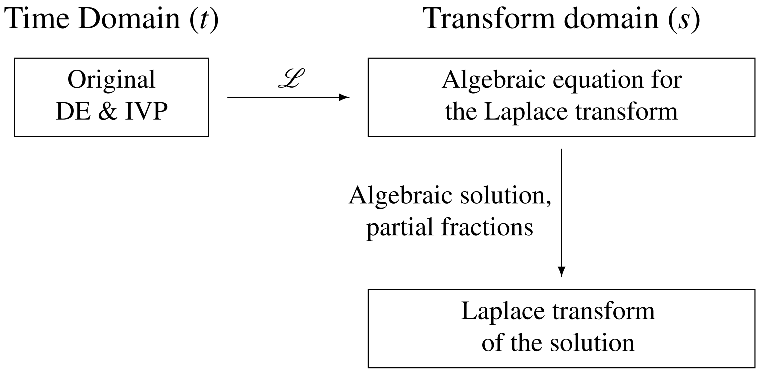
Laplace Transforms Turn Initial Value Problems into Algebra Problems in the s-Domain

Time Domain (t)		Transform domain (s)
Original DE & IVP	$\xrightarrow{\mathscr{L}}$	Algebraic equation for the Laplace transform

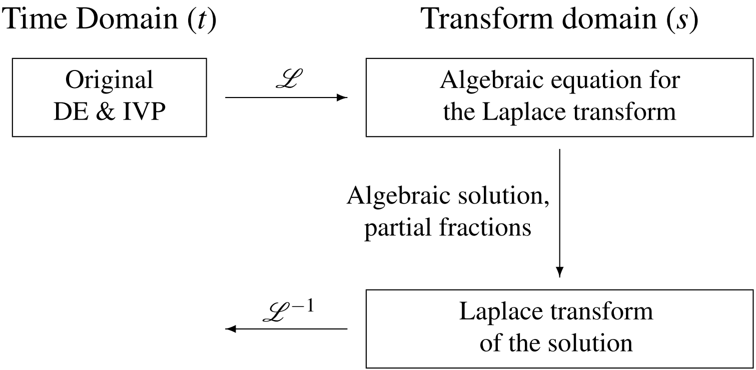
Laplace Transforms Turn Initial Value Problems into Algebra Problems in the s-Domain



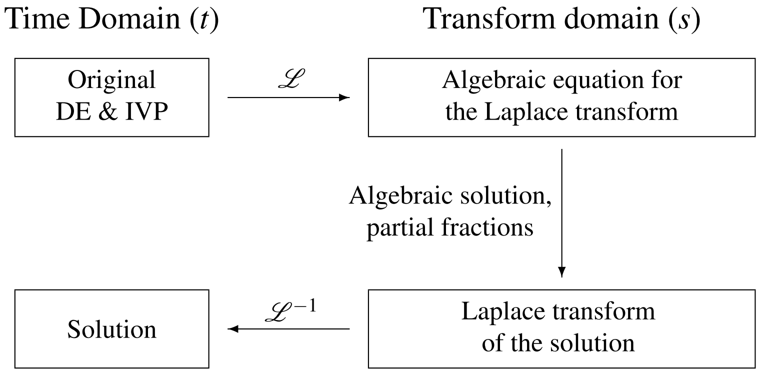
Laplace Transforms Turn Initial Value Problems into Algebra Problems in the *s*-Domain



Laplace Transforms Turn Initial Value Problems into Algebra Problems in the s-Domain



Laplace Transforms Turn Initial Value Problems into Algebra Problems in the s-Domain



The Laplace Transform of $f(t) = t$

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$$\mathscr{L}\{t\}(s)$$

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$$\begin{aligned}\mathscr{L}\{t\}(s) &= \int_0^\infty te^{-st}\,dt \\ &= t\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{t\to\infty}\end{aligned}$$

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\mathscr{L}\{t\}(s) &= \int_0^\infty te^{-st}\,dt \\
&= t\left(-\frac{1}{s}\right)e^{-st}\Big|_{t=0}^{t\to\infty} - \int_0^\infty \left(-\frac{1}{s}\right)e^{-st}\,dt \\
&= 0
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&= 0-0
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&= 0-0+\frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\left[\left(-\frac{1}{s}\right)e^{-st}\right]_{t=0}^{t\to\infty}
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But we don’t want to spend our life computing these integrals.

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But we don’t want to spend our life computing these integrals. Therefore, Laplace transforms are usually looked up in tables.

The Laplace Transform of $f(t) = 3t + e^{-4t}$

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$$\mathscr{L}\left\{3t + e^{-4t}\right\}(s)$$

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$$\mathscr{L}\left\{3t + e^{-4t}\right\}(s) = \mathscr{L}\left\{3t\right\}(s)$$

The Laplace Transform of $f(t) = 3t + e^{-4t}$

$$\mathscr{L}\left\{3t + e^{-4t}\right\}(s) \quad = \quad \mathscr{L}\{3t\}(s) + \mathscr{L}\left\{e^{-4t}\right\}(s)$$

The Laplace Transform of $f(t) = 3t + e^{-4t}$

$$
\begin{aligned}
\mathscr{L}\left\{3t + e^{-4t}\right\}(s) &= \mathscr{L}\{3t\}(s) + \mathscr{L}\left\{e^{-4t}\right\}(s) \\
&= 3\mathscr{L}\{t\}(s) + \mathscr{L}\left\{e^{-4t}\right\}(s)
\end{aligned}
$$

The Laplace Transform of $f(t) = 3t + e^{-4t}$

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\mathscr{L}\left\{3t + e^{-4t}\right\}(s) &= \mathscr{L}\{3t\}(s) + \mathscr{L}\left\{e^{-4t}\right\}(s) \\
&= 3\mathscr{L}\{t\}(s) + \mathscr{L}\left\{e^{-4t}\right\}(s) \\
&= 3\frac{1}{s^2}
\end{aligned}$$

The Laplace Transform of $f(t) = 3t + e^{-4t}$

$$\begin{aligned} \mathscr{L}\left\{3t+e^{-4t}\right\}(s) &= \mathscr{L}\{3t\}(s)+\mathscr{L}\left\{e^{-4t}\right\}(s) \\ &= 3\mathscr{L}\{t\}(s)+\mathscr{L}\left\{e^{-4t}\right\}(s) \\ &= 3\frac{1}{s^2}+\frac{1}{s+4} \end{aligned}$$

How Do We Get Back From the Transform Domain?

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How Do We Get Back From the Transform Domain?

1. There is a function $\mathscr{L}^{-1}$ that maps every Laplace transform back to the original function. Sensibly, it is called the **inverse Laplace transform**.
2. Similar to Laplace transforms, we have $\mathscr{L}^{-1}\{aF+bG\}=a\mathscr{L}^{-1}\{F\}+b\mathscr{L}^{-1}\{G\}$
3. Because many transforms are rational functions, inverting Laplace transforms involves lots of partial fraction decompositions.

The Inverse Laplace Transform of $F(s) = \frac{1}{s^2 + 8s + 15}$

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$$\frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} = \frac{A}{s+3}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} = \frac{A}{s+3}+\frac{B}{s+5}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} = \frac{A}{s+3}+\frac{B}{s+5}$$

$$1$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\begin{aligned}\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5}\\ 1 &= A(s+5)\end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} = \frac{A}{s+3}+\frac{B}{s+5}$$

$$1 = A(s+5)+B(s+3)$$

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$$1 = A(s+5)+B(s+3)$$

$$s=-3:$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

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$$1 = A(s+5)+B(s+3)$$

$$s=-3: \qquad 1$$

The Inverse Laplace Transform of $F(s) = \frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$1 = A(s+5) + B(s+3)$$

$$s=-3: \quad 1 = A \cdot 2$$

The Inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$1 = A(s+5) + B(s+3)$$

$$s=-3: \quad 1 = A\cdot 2 + B\cdot 0$$

The Inverse Laplace Transform of $F(s) = \dfrac{1}{s^2+8s+15}$

$$\begin{aligned} \frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)} &= \frac{A}{s+3} + \frac{B}{s+5} \\ 1 &= A(s+5)+B(s+3) \\ s=-3: \qquad 1 &= A\cdot 2 + B\cdot 0, \qquad A = \frac{1}{2} \end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\begin{aligned}
\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5}\\
1 &= A(s+5)+B(s+3)\\
s=-3: \qquad 1 &= A\cdot 2+B\cdot 0, \qquad A=\frac{1}{2}\\
s=-5: \qquad &
\end{aligned}$$

The Inverse Laplace Transform of $F(s) = \frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

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The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

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s=-3: \qquad 1 &= A\cdot 2+B\cdot 0, \qquad A=\frac{1}{2}\\
s=-5: \qquad 1 &= A\cdot 0
\end{aligned}$$

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\end{aligned}$$

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s=-3: \qquad 1 &= A\cdot 2 + B\cdot 0, \qquad A = \frac{1}{2} \\
s=-5: \qquad 1 &= A\cdot 0 + B\cdot(-2), \qquad B = -\frac{1}{2} \\
\frac{1}{s^2+8s+15} &
\end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$
\begin{aligned}
\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5} \\
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s=-3: \qquad 1 &= A\cdot 2+B\cdot 0, \qquad A=\frac{1}{2} \\
s=-5: \qquad 1 &= A\cdot 0+B\cdot(-2), \qquad B=-\frac{1}{2} \\
\frac{1}{s^2+8s+15} &= \frac{1}{2}
\end{aligned}
$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\begin{aligned}
\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5}\\
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s=-3: \qquad 1 &= A\cdot 2+B\cdot 0, \qquad A=\frac{1}{2}\\
s=-5: \qquad 1 &= A\cdot 0+B\cdot(-2), \qquad B=-\frac{1}{2}\\
\frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3}
\end{aligned}$$

The Inverse Laplace Transform of $F(s) = \frac{1}{s^2+8s+15}$

$$\begin{aligned} \frac{1}{s^2+8s+15} = \frac{1}{(s+3)(s+5)} &= \frac{A}{s+3} + \frac{B}{s+5} \\ 1 &= A(s+5) + B(s+3) \\ s=-3: \qquad 1 &= A\cdot 2 + B\cdot 0, \qquad A = \frac{1}{2} \\ s=-5: \qquad 1 &= A\cdot 0 + B\cdot(-2), \qquad B = -\frac{1}{2} \\ \frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3} - \frac{1}{2} \end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\begin{aligned}
\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5} \\
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s=-3: \qquad 1 &= A\cdot 2+B\cdot 0, \qquad A=\frac{1}{2} \\
s=-5: \qquad 1 &= A\cdot 0+B\cdot(-2), \qquad B=-\frac{1}{2} \\
\frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+5}
\end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} = \frac{A}{s+3}+\frac{B}{s+5}$$

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$$\frac{1}{s^2+8s+15} = \frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+5}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\}$$

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\frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+5} \\
\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} &= \frac{1}{2}
\end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

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\frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+5} \\
\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} &= \frac{1}{2}e^{-3t}
\end{aligned}$$

The Inverse Laplace Transform of $F(s)=\frac{1}{s^2+8s+15}$

$$
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\frac{1}{s^2+8s+15}=\frac{1}{(s+3)(s+5)} &= \frac{A}{s+3}+\frac{B}{s+5}\\
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\frac{1}{s^2+8s+15} &= \frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+5}\\
\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} &= \frac{1}{2}e^{-3t}-\frac{1}{2}
\end{aligned}
$$

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\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} &= \frac{1}{2}e^{-3t}-\frac{1}{2}e^{-5t}
\end{aligned}$$