Laplace Transforms of Step Functions

Bernd Schröder

Everything Remains As It Was

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Original DE & IVP

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Time Domain (t)

Original DE & IVP	$\xrightarrow{\mathscr{L}}$	Algebraic equation for the Laplace transform

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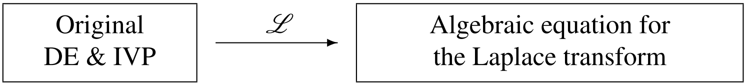
Time Domain (t) Transform domain (s)



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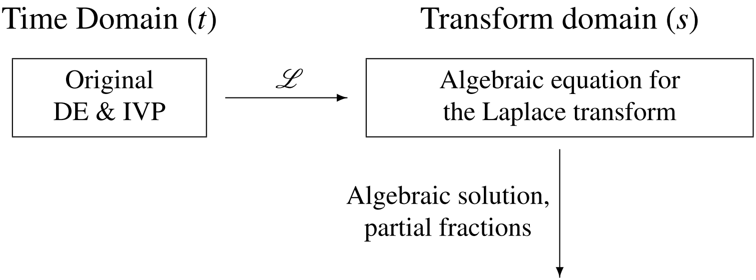

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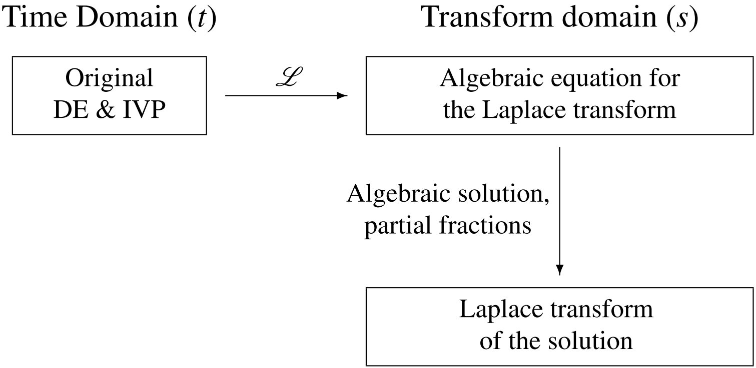

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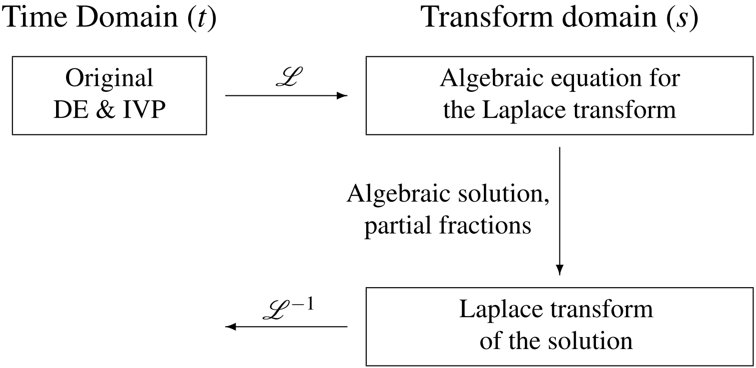

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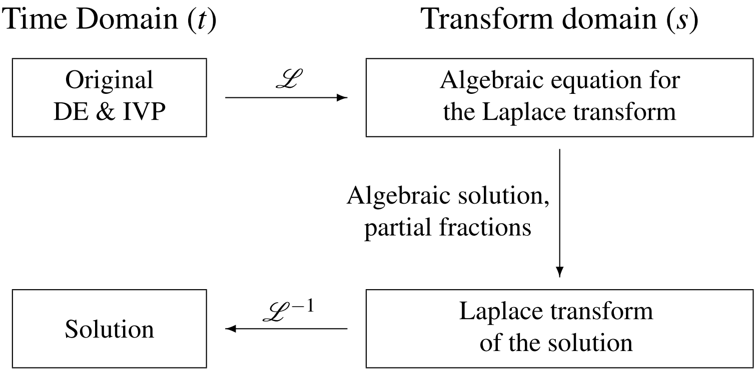

The Unit Step Function

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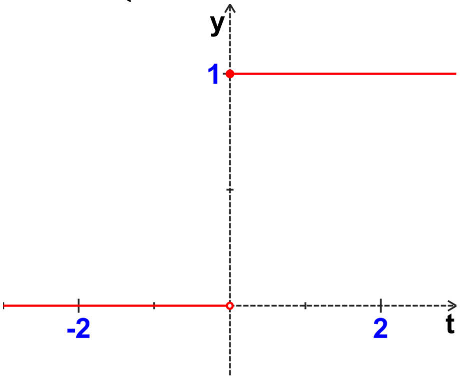

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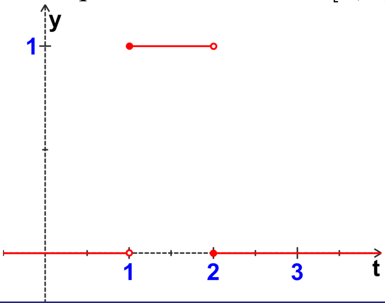

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3. The function $\mathscr{U}(t-a) - \mathscr{U}(t-b)$ is equal to 1 on $[a,b)$ and equal to zero outside $[a,b)$.
4. The function $f(t)\mathscr{U}(t-a) - f(t)\mathscr{U}(t-b)$ is equal to f on $[a,b)$ and equal to zero outside $[a,b)$.

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5. $\mathscr{L}\left\{f(t-a)\mathscr{U}(t-a)\right\}(s) = e^{-as}F(s)$.
6. Keep the exponential separate when working in the transform domain.

An Application Problem

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(Dimensions fictitious.)

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In an RC circuit with resistance $R = 1\Omega$ and capacitance $C = \frac{1}{3}F$ initially, the charge of the capacitor is $2C$. At time $t = 2\pi$ seconds, a sine shaped external voltage is activated. At time $t = 5\pi$ seconds, the external voltage is turned off.

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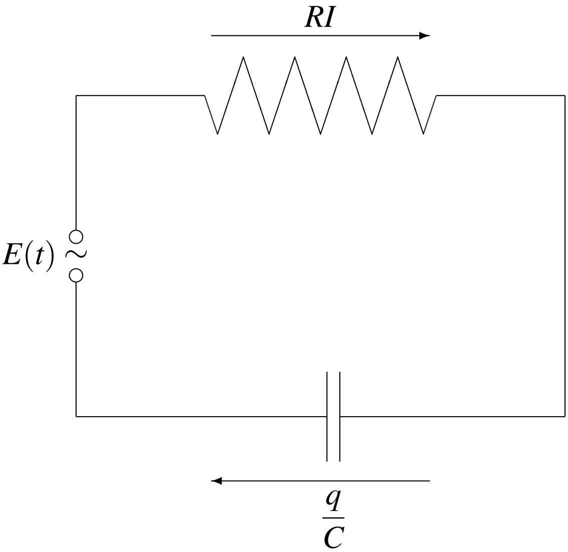

$$E(t) = Rq' + \frac{1}{C}q$$

Underlying Equations

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- $E(t) = \sin(t)\mathscr{U}(t - 2\pi)$

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- $y(0) = 2$
- $E(t)$ is $\sin(t)$, activated at $t = 2\pi$ and deactivated at $t = 5\pi$.
- $E(t) = \sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$

Solve the Initial Value Problem

$y' + 3y = \sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$,
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Adjusting the right side.

Solve the Initial Value Problem $y' + 3y = \sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$, $y(0) = 2$.

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$$\sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$$

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Adjusting the right side.

$$\begin{aligned} &\sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi) \\ &= \sin(t-2\pi+2\pi)\mathscr{U}(t-2\pi) - \sin(t-5\pi+5\pi)\mathscr{U}(t-5\pi) \end{aligned}$$

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$$\begin{aligned}
&\sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi) \\
&= \sin\big((t-2\pi)+2\pi\big)\mathscr{U}(t-2\pi) - \sin(t-5\pi+5\pi)\mathscr{U}(t-5\pi)
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&= \sin(t-2\pi)\mathscr{U}(t-2\pi)
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$$sY - 2$$

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$$sY - 2 + 3Y$$

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$$\begin{aligned} y' + 3y &= \sin(t-2\pi)\mathscr{U}(t-2\pi) + \sin(t-5\pi)\mathscr{U}(t-5\pi), \\ & \qquad y(0) = 2 \\ sY - 2 + 3Y &= e^{-2\pi s}\frac{1}{s^2+1} \end{aligned}$$

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& \qquad y(0) = 2 \\
sY - 2 + 3Y &= e^{-2\pi s}\frac{1}{s^2+1} + e^{-5\pi s}\frac{1}{s^2+1} \\
(s+3)Y - 2 &= e^{-2\pi s}\frac{1}{s^2+1} + e^{-5\pi s}\frac{1}{s^2+1} \\
(s+3)Y &= e^{-2\pi s}\frac{1}{s^2+1} + e^{-5\pi s}\frac{1}{s^2+1} + 2 \\
Y &= e^{-2\pi s}\frac{1}{(s^2+1)(s+3)} + e^{-5\pi s}\frac{1}{(s^2+1)(s+3)} + \frac{2}{s+3}
\end{aligned}
$$

Partial Fraction Decomposition.

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$$\frac{1}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1}$$

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$$1 = (As+B)(s+3)+C\left(s^2+1\right)$$

Partial Fraction Decomposition.

$$\begin{aligned} \frac{1}{(s^2+1)(s+3)} &= \frac{As+B}{s^2+1}+\frac{C}{s+3} \\ 1 &= (As+B)(s+3)+C\left(s^2+1\right) \end{aligned}$$

$s=-3:$

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$$s=1:$$

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s=-3: \qquad & 1=10C, \qquad C=\frac{1}{10} \\
s=0: \qquad & 1=3B+\frac{1}{10}, \qquad B=\frac{3}{10} \\
s=1: \qquad & 1=\left(A+\frac{3}{10}\right)4+\frac{1}{10}\cdot 2
\end{aligned}$$

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\frac{1}{\left(s^2+1\right)(s+3)} &= \frac{As+B}{s^2+1}+\frac{C}{s+3} \\
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$$\frac{1}{(s^2+1)(s+3)} = \frac{1}{10}\frac{-s+3}{s^2+1}$$

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$$s=1: \qquad 1 = \left(A + \frac{3}{10}\right)4 + \frac{1}{10}\cdot 2, \qquad A = -\frac{1}{10}$$

$$\frac{1}{(s^2+1)(s+3)} = \frac{1}{10}\frac{-s+3}{s^2+1} + \frac{1}{10}\frac{1}{s+3}$$

Inverting the Laplace transform.

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$$\begin{aligned} Y &= e^{-2\pi s}\left(\frac{1}{10}\frac{-s+3}{s^2+1}+\frac{1}{10}\frac{1}{s+3}\right) \\ &\quad +e^{-5\pi s}\left(\frac{1}{10}\frac{-s+3}{s^2+1}+\frac{1}{10}\frac{1}{s+3}\right)+\frac{2}{s+3} \end{aligned}$$

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\begin{aligned}
Y &= e^{-2\pi s}\left(\frac{1}{10}\frac{-s+3}{s^2+1}+\frac{1}{10}\frac{1}{s+3}\right) \\
&\quad +e^{-5\pi s}\left(\frac{1}{10}\frac{-s+3}{s^2+1}+\frac{1}{10}\frac{1}{s+3}\right)+\frac{2}{s+3} \\
&= \frac{1}{10}e^{-2\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right) \\
&\quad +\frac{1}{10}e^{-5\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right)+2\frac{1}{s+3} \\
y &= \frac{1}{10}
\end{aligned}
$$

Inverting the Laplace transform.

$$
\begin{aligned}
Y &= e^{-2\pi s}\left(\frac{1}{10}\frac{-s+3}{s^2+1}+\frac{1}{10}\frac{1}{s+3}\right) \\
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&= \frac{1}{10}e^{-2\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right) \\
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y &= \frac{1}{10}\mathscr{U}(t-2\pi)
\end{aligned}
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&= \frac{1}{10}e^{-2\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right)\\
&\quad +\frac{1}{10}e^{-5\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right)+2\frac{1}{s+3}\\
y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)
\end{aligned}
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&\quad +\frac{1}{10}e^{-5\pi s}\left(-\frac{s}{s^2+1}+3\frac{1}{s^2+1}+\frac{1}{s+3}\right)+2\frac{1}{s+3}\\
y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-2\pi}
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&\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-5\pi}
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&\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-5\pi}+2e^{-3t}
\end{aligned}
$$

Solve the Initial Value Problem

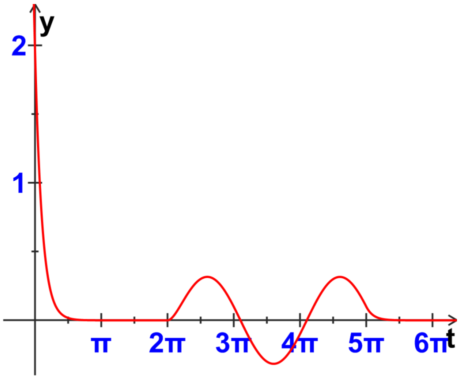
$y' + 3y = \sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$,
$y(0) = 2$.

$$\begin{aligned} y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-2\pi} \\ &\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-5\pi}+2e^{-3t} \end{aligned}$$

Solve the Initial Value Problem $y' + 3y = \sin(t)\mathscr{U}(t-2\pi) - \sin(t)\mathscr{U}(t-5\pi)$, $y(0) = 2$.

$$\begin{aligned} y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-2\pi} \\ &\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(-\cos(t)+3\sin(t)+e^{-3t}\right)_{t\to t-5\pi}+2e^{-3t} \\ &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3(t-2\pi)}\right) \\ &\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(\cos(t)-3\sin(t)+e^{-3(t-5\pi)}\right)+2e^{-3t} \end{aligned}$$

$$y = \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3(t-2\pi)}\right)$$
$$+\frac{1}{10}\mathscr{U}(t-5\pi)\left(\cos(t)-3\sin(t)+e^{-3(t-5\pi)}\right)+2e^{-3t}$$



Checking the Solution.

Checking the Solution.

$$
\begin{aligned}
y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3(t-2\pi)}\right) \\
&\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(\cos(t)-3\sin(t)+e^{-3(t-5\pi)}\right)+2e^{-3t}
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- Initial value: “By inspection.”
- The function $y = e^{-3t}$ solves the differential equation $y' + 3y = 0$.

Checking the Solution.

$$\begin{aligned} y &= \frac{1}{10}\mathscr{U}(t-2\pi)\left(-\cos(t)+3\sin(t)+e^{-3(t-2\pi)}\right) \\ &\quad +\frac{1}{10}\mathscr{U}(t-5\pi)\left(\cos(t)-3\sin(t)+e^{-3(t-5\pi)}\right)+2e^{-3t} \end{aligned}$$

- Initial value: “By inspection.”
- The function $y = e^{-3t}$ solves the differential equation $y' + 3y = 0$.
- So all exponential terms in the solution are o.k., provided that the rest, which is $\frac{1}{10}\left(\mathscr{U}(t-2\pi)-\mathscr{U}(t-5\pi)\right)\left(-\cos(t)+3\sin(t)\right)$ produces a sine function that only exists on $[2\pi, 5\pi)$.

Checking the Solution.

Checking the Solution.

$$\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)'+3\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)$$

Checking the Solution.

$$\begin{aligned}
&\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)'+3\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)\\
&= \frac{1}{10}\big(\sin(t)+3\cos(t)\big)+3\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)
\end{aligned}$$

Checking the Solution.

$$
\begin{aligned}
&\frac{1}{10}\Big(-\cos(t)+3\sin(t)\Big)'+3\frac{1}{10}\Big(-\cos(t)+3\sin(t)\Big)\\
&= \frac{1}{10}\Big(\sin(t)+3\cos(t)\Big)+3\frac{1}{10}\Big(-\cos(t)+3\sin(t)\Big)\\
&= \frac{1}{10}(3-3)\cos(t)+\frac{1}{10}(1+9)\sin(t)
\end{aligned}
$$

Checking the Solution.

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&= \sin(t)
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&\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)'+3\frac{1}{10}\big(-\cos(t)+3\sin(t)\big)\\
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$$