Taylor Polynomials

Bernd Schröder

Linear and Higher Order Approximations

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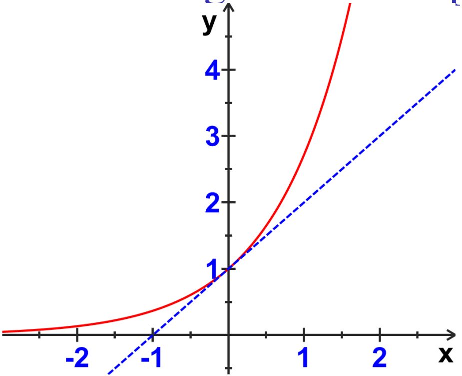
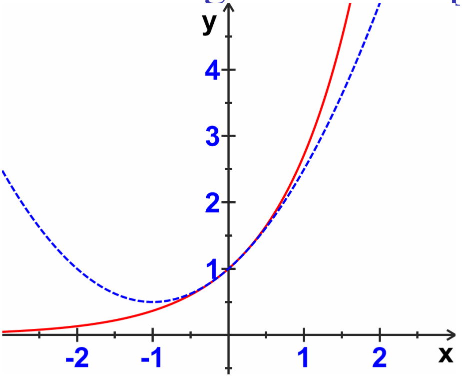
2. This is because the linear approximation's value and its first derivative agree with those of the function at a.
3. We should get better approximations with functions that match more derivatives of f at a.

Linear and Higher Order Approximations



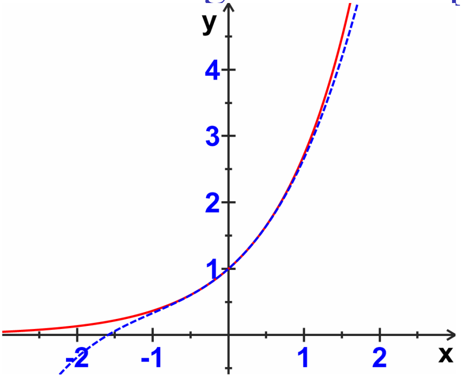
$N = 1$

Linear and Higher Order Approximations



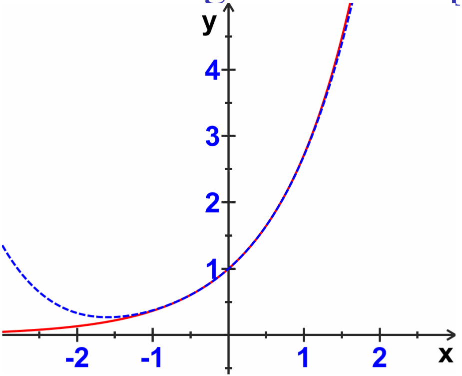
$N = 2$

Linear and Higher Order Approximations



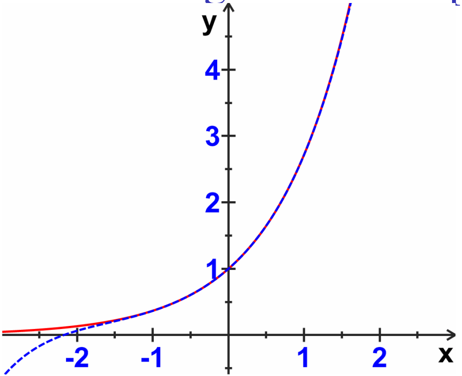
$N = 3$

Linear and Higher Order Approximations



$N = 4$

Linear and Higher Order Approximations



$N = 5$

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4. The polynomial

$$\begin{aligned} T_N(x) &:= \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(N)}(a)}{N!}(x-a)^N \end{aligned}$$

is called the N^{th} **Taylor polynomial** of f at a.

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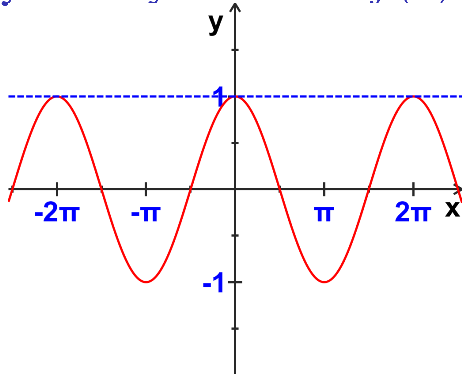
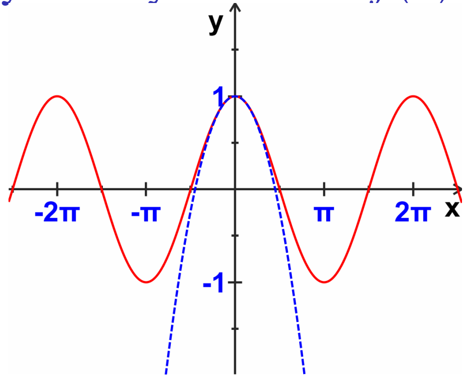
Taylor Polynomials of $f(x) = \cos(x)$



$N = 0$

$$T_0(x) = \sum_{n=0}^{0} \frac{(-1)^n}{(2n)!} x^{2n} = 1$$

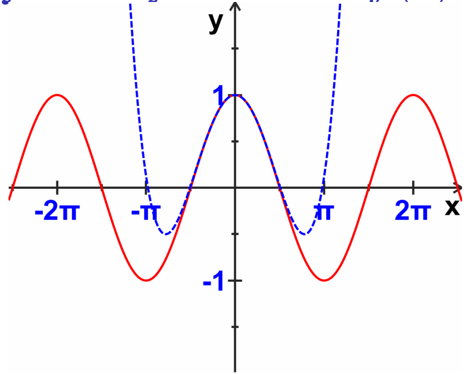
Taylor Polynomials of $f(x) = \cos(x)$



$N = 2$

$$T_2(x) = \sum_{n=0}^{1} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2}$$

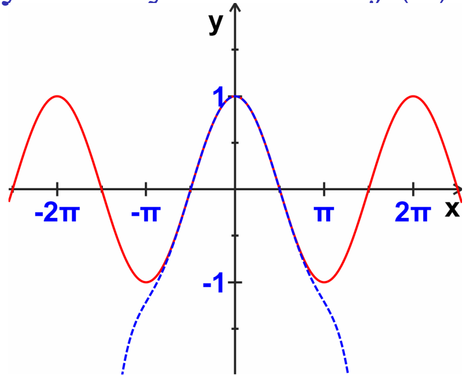
Taylor Polynomials of $f(x) = \cos(x)$



$N = 4$

$$T_4(x) = \sum_{n=0}^{2} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

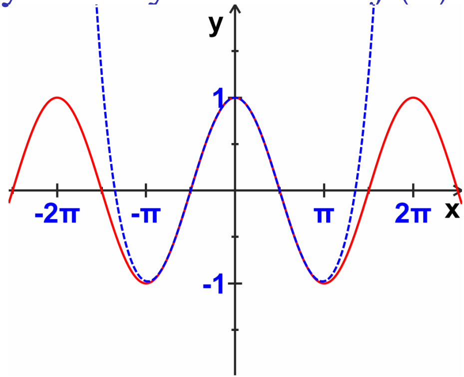
Taylor Polynomials of $f(x) = \cos(x)$



$N = 6$

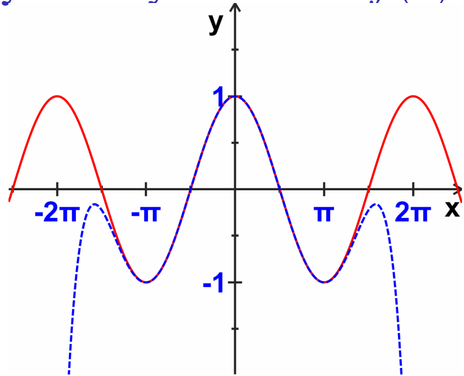
$$T_6(x) = \sum_{n=0}^{3} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

Taylor Polynomials of $f(x) = \cos(x)$



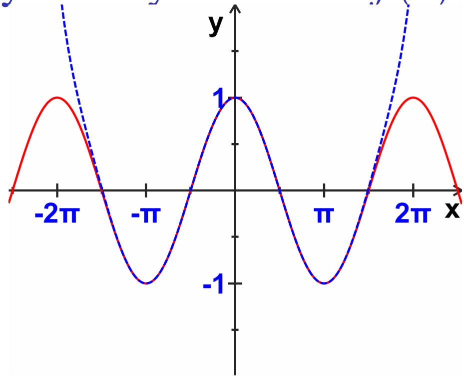
$N = 8$

Taylor Polynomials of $f(x) = \cos(x)$



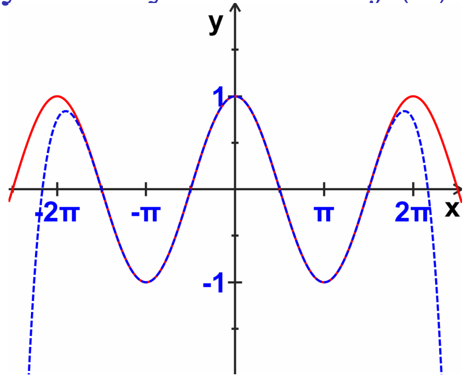
$N = 10$

Taylor Polynomials of $f(x) = \cos(x)$



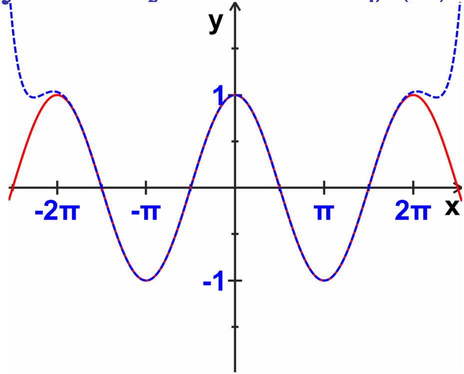
$N = 12$

Taylor Polynomials of $f(x) = \cos(x)$



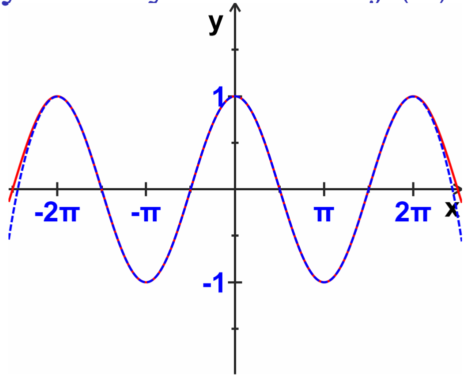
$N = 14$

Taylor Polynomials of $f(x) = \cos(x)$



$N = 16$

Taylor Polynomials of $f(x) = \cos(x)$



$N = 18$