The Heat Equation

Bernd Schröder

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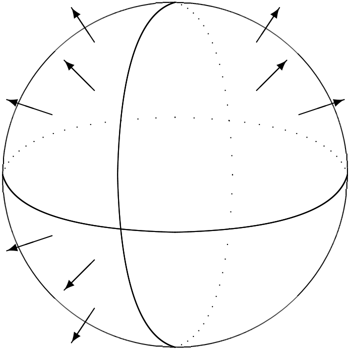

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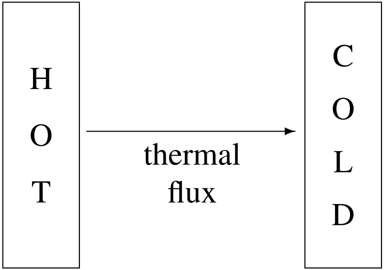

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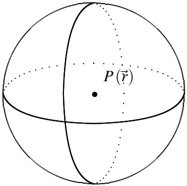

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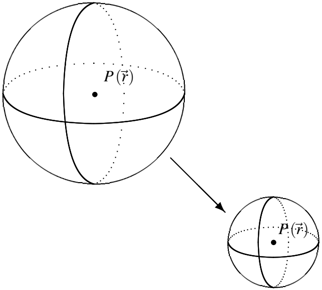

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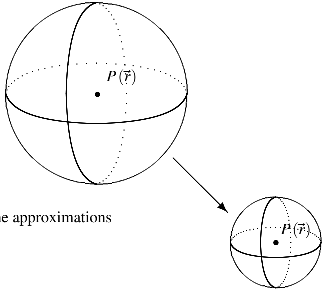

As the radius a shrinks, the approximations

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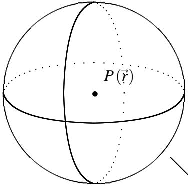
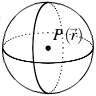


As the radius a shrinks, the approximations

$$\iiint_B \operatorname{div}(\operatorname{grad}(u))\ dV \approx \frac{4}{3}\pi a^3 \operatorname{div}(\operatorname{grad}(u)),\ \iiint_B k\frac{\partial u}{\partial t}\ dV \approx \frac{4}{3}\pi a^3 k\frac{\partial u}{\partial t}$$



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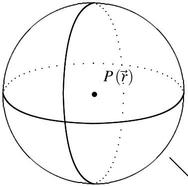

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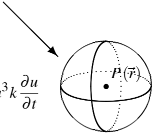
improve towards equality.



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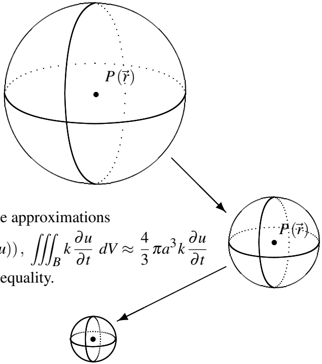

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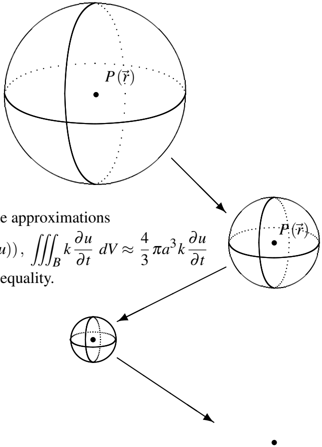

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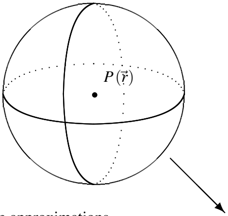

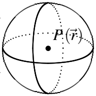

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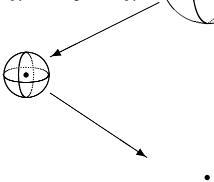

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