Homogeneous First Order Equations

Bernd Schröder

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That's it.

Solve the Initial Value Problem $y' = \frac{y}{x} - e^{\frac{y}{x}}$, $y(1) = 1$.

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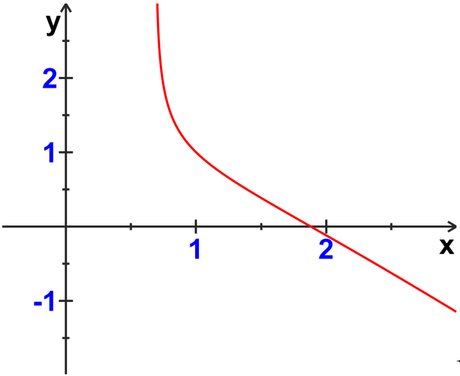
\ln(-c) &= -1 \\
-c &= e^{-1} \\
c &= -e^{-1} \\
y &= -x\ln\left(\ln|x| + \frac{1}{e}\right)
\end{aligned}$$

Solve the Initial Value Problem $y' = \frac{y}{x} - e^{\frac{y}{x}}$, $y(1) = 1$.



$$y = -x \ln\left(\ln|x| + \frac{1}{e} \right)$$

Does $y=-x\ln\left(\ln|x|+\frac{1}{e}\right)$ Really Solve the Initial Value Problem $y'=\frac{y}{x}-e^{\frac{y}{x}}$, $y(1)=1$?

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