The Method of Frobenius

Bernd Schröder
What is the Method of Frobenius?

The method of Frobenius works for differential equations of the form:

\[ y'' + P(x)y' + Q(x)y = 0 \]

in which \( P \) or \( Q \) is not analytic at the point of expansion \( x_0 \).

But \( P \) and \( Q \) cannot be arbitrary:

\[ (x - x_0)P(x) \text{ and } (x - x_0)^2Q(x) \]

must be analytic at \( x_0 \).

Instead of a series solution

\[ y = \sum_{n=0}^{\infty} c_n(x - x_0)^n \]

we obtain a solution of the form

\[ y = \sum_{n=0}^{\infty} c_n(x - x_0)^n + r \]

The method of Frobenius is guaranteed to produce one solution, but it may not produce two linearly independent solutions.
What is the Method of Frobenius?

1. The method of Frobenius works for differential equations of the form $y'' + P(x)y' + Q(x)y = 0$ in which $P$ or $Q$ is not analytic at the point of expansion $x_0$. 
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2. But $P$ and $Q$ cannot be arbitrary: $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ must be analytic at $x_0$. 
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2. But $P$ and $Q$ cannot be arbitrary: $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ must be analytic at $x_0$.

3. Instead of a series solution $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, we obtain a solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$. 

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2. But \( P \) and \( Q \) cannot be arbitrary: \((x - x_0)P(x)\) and \((x - x_0)^2Q(x)\) must be analytic at \( x_0 \).

3. Instead of a series solution \( y = \sum_{n=0}^{\infty} c_n (x - x_0)^n \), we obtain a solution of the form \( y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \).

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5. As for series solutions, we substitute the series and its derivatives into the equation to obtain an equation for $r$ and a set of equations for the $c_n$. 
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6. These equations will allow us to compute $r$ and the $c_n$. 
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That’s it.
Solve the Differential Equation

$$9x^2 y'' + 3x^2 y' + 2y = 0$$
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\[ 9x^2 \left( \sum_{n=0}^{\infty} c_n x^{n+r} \right)'' + 3x^2 \left( \sum_{n=0}^{\infty} c_n x^{n+r} \right)' + 2 \left( \sum_{n=0}^{\infty} c_n x^{n+r} \right) = 0 \]
Solve the Differential Equation

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Solve the Differential Equation

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Solve the Differential Equation

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9x^2 \sum_{n=0}^{\infty} c_n (n + r)(n + r - 1)x^{n+r-2} + 3x^2 \sum_{n=0}^{\infty} c_n (n + r)x^{n+r-1} + 2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0
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\]

\[
\sum_{n=0}^{\infty} 9(n + r)(n + r - 1)c_nx^{n+r} + \sum_{n=0}^{\infty} 3(n + r)c_nx^{n+r+1} + \sum_{n=0}^{\infty} 2c_nx^{n+r} = 0
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Solve the Differential Equation

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\]

\[
\sum_{k=0}^{\infty} 9(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=1}^{\infty} 3(k+r-1)c_{k-1} x^{k+r} + \sum_{k=0}^{\infty} 2c_k x^{k+r} = 0
\]
Solve the Differential Equation

\[ 9x^2y''' + 3x^2y' + 2y = 0 \]

\[
9x^2 \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + 3x^2 \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + 2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0
\]

\[
\sum_{n=0}^{\infty} 9(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0
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\sum_{k=0}^{\infty} 9(k+r)(k+r-1)c_k x^{k+r} + \sum_{k=0}^{\infty} 3(k+r-1)c_{k-1} x^{k+r} + \sum_{k=0}^{\infty} 2c_k x^{k+r} = 0
\]

\[
(9r(r-1)+2)c_0 x^r + \sum_{k=1}^{\infty} \left[ 9(k+r)(k+r-1)c_k + 3(k+r-1)c_{k-1} + 2c_k \right] x^{k+r} = 0
\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

Indicial equation:

\[ (9r(r - 1) + 2)c_0 = 0 \]
Solve the Differential Equation

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Indicial equation:

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\[9r(r - 1) + 2 = 0\]

\[9r^2 - 9r + 2 = 0\]
Solve the Differential Equation

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Indicial equation:

\[
(9r(r - 1) + 2)c_0 = 0 \\
9r(r - 1) + 2 = 0 \\
9r^2 - 9r + 2 = 0 \\
r_{1,2} = \frac{9 \pm \sqrt{81 - 72}}{18}
\]
Solve the Differential Equation

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Indicial equation:

\[
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\]

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9r(r - 1) + 2 = 0
\]

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9r^2 - 9r + 2 = 0
\]

\[
r_{1,2} = \frac{9 \pm \sqrt{81 - 72}}{18}
\]

\[
= \frac{9 \pm 3}{18}
\]
Solve the Differential Equation

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Indicial equation:

\[
\begin{align*}
(9r(r - 1) + 2)c_0 & = 0 \\
9r(r - 1) + 2 & = 0 \\
9r^2 - 9r + 2 & = 0 \\
\end{align*}
\]

\[
\begin{align*}
r_{1,2} & = \frac{9 \pm \sqrt{81 - 72}}{18} \\
& = \frac{9 \pm 3}{18} \\
& = \frac{2}{3}
\end{align*}
\]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

Indicial equation:

$$(9r(r-1) + 2)c_0 = 0$$
$$9r(r-1) + 2 = 0$$
$$9r^2 - 9r + 2 = 0$$

$$r_{1,2} = \frac{9 \pm \sqrt{81 - 72}}{18}$$
$$= \frac{9 \pm 3}{18}$$
$$= \frac{2}{3}, \frac{1}{3}$$
Solve the Differential Equation
\[ 9x^2y'' + 3x^2y' + 2y = 0 \]
Recurrence Relation.
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

Recurrence Relation.

\[ 9(k + r)(k + r - 1)c_k + 3(k + r - 1)c_{k-1} + 2c_k = 0 \]
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

Recurrence Relation.

\[
\begin{align*}
9(k+r)(k+r-1)c_k + 3(k+r-1)c_{k-1} + 2c_k &= 0 \\
[9(k+r)(k+r-1) + 2]c_k + 3(k+r-1)c_{k-1} &= 0
\end{align*}
\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

Recurrence Relation.

\[
9(k+r)(k+r-1)c_k + 3(k+r-1)c_{k-1} + 2c_k = 0
\]

\[
\left[ 9(k+r)(k+r-1) + 2 \right] c_k + 3(k+r-1)c_{k-1} = 0
\]

\[
\left[ 9(k+r)(k+r-1) + 2 \right] c_k = -3(k+r-1)c_{k-1}
\]
Solve the Differential Equation

\[9x^2y'' + 3x^2y' + 2y = 0\]

Recurrence Relation.

\[9(k + r)(k + r - 1)c_k + 3(k + r - 1)c_{k-1} + 2c_k = 0\]

\[\left[9(k + r)(k + r - 1) + 2\right]c_k + 3(k + r - 1)c_{k-1} = 0\]

\[\left[9(k + r)(k + r - 1) + 2\right]c_k = -3(k + r - 1)c_{k-1}\]

\[c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2}c_{k-1}\]

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\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{2}{3} \]
Solve the Differential Equation

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\[r = \frac{2}{3}, \quad c_0 = 1\]
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\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left(k + \frac{2}{3}\right) 3 \left(k + \frac{2}{3} - 1\right) + 2} c_{k-1} \]
Solve the Differential Equation

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Solve the Differential Equation

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Solve the Differential Equation

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\[= \frac{1 - 3k}{(3k + 2)(3k - 1) + 2}c_{k-1} = \frac{1 - 3k}{9k^2 + 3k}c_{k-1}\]

\[c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1}c_{1-1}\]
Solve the Differential Equation

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\[ r = \frac{2}{3}, \ c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left(k + \frac{2}{3}\right) 3 \left(k + \frac{2}{3} - 1\right) + 2} c_{k-1} \]

\[ = \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_{k-1} \]

\[ c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_{1-1} = -\frac{2}{12} \cdot 1 \]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{2}{3}, \quad c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left( k + \frac{2}{3} \right) 3 \left( k + \frac{2}{3} - 1 \right) + 2} c_{k-1} \]

\[ = \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_{k-1} \]

\[ c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6} \]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{2}{3}, \ c_0 = 1$$

$$c_k = \frac{3 - 3k - 3r}{9(k+r)(k+r-1)+2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3(k+\frac{2}{3})3(k+\frac{2}{3}-1)+2}c_{k-1}$$

$$= \frac{1 - 3k}{(3k+2)(3k-1)+2}c_{k-1} = \frac{1 - 3k}{9k^2+3k}c_{k-1}$$

$$c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2+3 \cdot 1}c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6}$$

$$c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2+3 \cdot 2}c_{2-1}$$
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{2}{3}, \quad c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left( k + \frac{2}{3} \right) 3 \left( k + \frac{2}{3} - 1 \right) + 2}c_{k-1} \]

\[ = \frac{1 - 3k}{(3k + 2)(3k - 1) + 2}c_{k-1} = \frac{1 - 3k}{9k^2 + 3k}c_{k-1} \]

\[ c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1}c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6} \]

\[ c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2 + 3 \cdot 2}c_{2-1} = -\frac{5}{42} \left( -\frac{1}{6} \right) \]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{2}{3}, \quad c_0 = 1$$

$$c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3(k + \frac{2}{3}) 3(k + \frac{2}{3} - 1) + 2} c_{k-1}$$

$$= \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_{k-1}$$

$$c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6}$$

$$c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2 + 3 \cdot 2} c_{2-1} = -\frac{5}{42} \left(-\frac{1}{6}\right) = \frac{5}{252}$$
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{2}{3}, \quad c_0 = 1$$

$$c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left(k + \frac{2}{3}\right) 3 \left(k + \frac{2}{3} - 1\right) + 2} c_{k-1}$$

$$= \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_{k-1}$$

$$c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6}$$

$$c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2 + 3 \cdot 2} c_{2-1} = -\frac{5}{42} \left(-\frac{1}{6}\right) = \frac{5}{252}$$

$$c_3 = \frac{1 - 3 \cdot 3}{9 \cdot 3^2 + 3 \cdot 3} c_{3-1}$$
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{2}{3}, \ c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_k^{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3 \left( k + \frac{2}{3} \right) 3 \left( k + \frac{2}{3} - 1 \right) + 2} c_k^{k-1} \]

\[ = \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_k^{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_k^{k-1} \]

\[ c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_1^{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6} \]

\[ c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2 + 3 \cdot 2} c_2^{2-1} = -\frac{5}{42} \left( -\frac{1}{6} \right) = \frac{5}{252} \]

\[ c_3 = \frac{1 - 3 \cdot 3}{9 \cdot 3^2 + 3 \cdot 3} c_3^{3-1} = -\frac{8}{90} \cdot \frac{5}{252} \]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{2}{3}, \; c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{2}{3}}{3(k + \frac{2}{3}) 3 \left(k + \frac{2}{3} - 1\right) + 2} c_{k-1} \]

\[ = \frac{1 - 3k}{(3k + 2)(3k - 1) + 2} c_{k-1} = \frac{1 - 3k}{9k^2 + 3k} c_{k-1} \]

\[ c_1 = \frac{1 - 3 \cdot 1}{9 \cdot 1^2 + 3 \cdot 1} c_{1-1} = -\frac{2}{12} \cdot 1 = -\frac{1}{6} \]

\[ c_2 = \frac{1 - 3 \cdot 2}{9 \cdot 2^2 + 3 \cdot 2} c_{2-1} = -\frac{5}{42} \left(-\frac{1}{6}\right) = \frac{5}{252} \]

\[ c_3 = \frac{1 - 3 \cdot 3}{9 \cdot 3^2 + 3 \cdot 3} c_{3-1} = -\frac{8}{90} \frac{5}{252} = -\frac{1}{567} \]
Solve the Differential Equation

$$9x^2 y'' + 3x^2 y' + 2y = 0$$

$$r = \frac{2}{3}$$

$$c_4 = \frac{1 - 3 \cdot 4}{9 \cdot 4^2 + 3 \cdot 4} c_4 - 1$$
Solve the Differential Equation
\[9x^2 y'' + 3x^2 y' + 2y = 0\]

\[r = \frac{2}{3}\]

\[c_4 = \frac{1 - 3 \cdot 4}{9 \cdot 4^2 + 3 \cdot 4} c_4 - 1 = -\frac{11}{156} \left(-\frac{1}{567}\right)\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{2}{3} \]

\[ c_4 = \frac{1 - 3 \cdot 4}{9 \cdot 4^2 + 3 \cdot 4}c_{4-1} = -\frac{11}{156} \left( -\frac{1}{567} \right) = \frac{11}{88452} \]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{2}{3}$$

$$c_4 = \frac{1 - 3 \cdot 4}{9 \cdot 4^2 + 3 \cdot 4} c_{4-1} = -\frac{11}{156} \left( -\frac{1}{567} \right) = \frac{11}{88452}$$

$$y_1 = x^\frac{2}{3} - \frac{1}{6} x^\frac{2}{3} + 1 + \frac{5}{252} x^\frac{2}{3} + 2 - \frac{1}{567} x^\frac{2}{3} + 3 + \frac{11}{88452} x^\frac{2}{3} + 4$$
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{2}{3}$$

$$c_4 = \frac{1 - 3 \cdot 4}{9 \cdot 4^2 + 3 \cdot 4} c_4 - 1 = -\frac{11}{156} \left( -\frac{1}{567} \right) = \frac{11}{88452}$$

$$y_1 = x^3 - \frac{1}{6} x^3 + \frac{5}{252} x^3 + 2 - \frac{1}{567} x^3 + 3 + \frac{11}{88452} x^3 + 4$$

$$= x^3 - \frac{1}{6} x^3 + \frac{5}{252} x^3 - \frac{1}{567} x^\frac{11}{3} + \frac{11}{88452} x^\frac{14}{3}$$
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{1}{3} \]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3}, \ c_0 = 1 \]
Solve the Differential Equation
\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]
\[
r = \frac{1}{3}, \ c_0 = 1
\]
\[
c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1}
\]
Solve the Differential Equation

\(9x^2y'' + 3x^2y' + 2y = 0\)

\(r = \frac{1}{3}, c_0 = 1\)

\[c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3(k + \frac{1}{3}) 3(k + \frac{1}{3} - 1) + 2} c_{k-1}\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3}, \quad c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left( k + \frac{1}{3} \right) 3 \left( k + \frac{1}{3} - 1 \right) + 2} c_{k-1} \]

\[ = \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} \]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{1}{3}, \ c_0 = 1$$

$$c_k = \frac{3 - 3k - 3r}{9(k+r)(k+r-1)+2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left( k + \frac{1}{3} \right) 3 \left( k + \frac{1}{3} - 1 \right) + 2}c_{k-1}$$

$$= \frac{2 - 3k}{(3k+1)(3k-2)+2}c_{k-1} = \frac{2 - 3k}{9k^2 - 3k}c_{k-1}$$
Solve the Differential Equation

\[9x^2y'' + 3x^2y' + 2y = 0\]

\[r = \frac{1}{3}, \ c_0 = 1\]

\[c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left( k + \frac{1}{3} \right) 3 \left( k + \frac{1}{3} - 1 \right) + 2} c_{k-1}\]

\[= \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} = \frac{2 - 3k}{9k^2 - 3k} c_{k-1}\]

\[c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1} c_{1-1}\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3} \], \( c_0 = 1 \)

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3(k + \frac{1}{3}) 3(k + \frac{1}{3} - 1) + 2} c_{k-1} \]

\[ = \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} = \frac{2 - 3k}{9k^2 - 3k} c_{k-1} \]

\[ c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1} c_{1-1} = -\frac{1}{6} \]
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{1}{3}, \quad c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left( k + \frac{1}{3} \right) 3 \left( k + \frac{1}{3} - 1 \right) + 2} c_{k-1} \]

\[ = \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} = \frac{2 - 3k}{9k^2 - 3k} c_{k-1} \]

\[ c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1} c_1 - 1 = \frac{1}{6} \]

\[ c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2} c_{2-1} \]
Solve the Differential Equation

$$9x^2 y'' + 3x^2 y' + 2y = 0$$

$$r = \frac{1}{3}, c_0 = 1$$

$$c_k = \frac{3 - 3k - 3r}{9(k+r)(k+r-1) + 2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3(k + \frac{1}{3}) 3(k + \frac{1}{3} - 1) + 2}c_{k-1}$$

$$= \frac{2 - 3k}{(3k+1)(3k-2) + 2}c_{k-1} = \frac{2 - 3k}{9k^2 - 3k}c_{k-1}$$

$$c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1}c_{1-1} = -\frac{1}{6}$$

$$c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2}c_{2-1} = -\frac{4}{30} \left( -\frac{1}{6} \right)$$
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3}, \quad c_0 = 1 \]

\[
    c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left(k + \frac{1}{3}\right) 3 \left(k + \frac{1}{3} - 1\right) + 2} c_{k-1} \\
    = \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} = \frac{2 - 3k}{9k^2 - 3k} c_{k-1} \\
    c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1} c_{1-1} = -\frac{1}{6} \\
    c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2} c_{2-1} = -\frac{4}{30} \left(-\frac{1}{6}\right) = \frac{1}{45} \]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3}, \quad c_0 = 1 \]

\[ c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2} c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3(k + \frac{1}{3})3(k + \frac{1}{3} - 1) + 2} c_{k-1} \]

\[ = \frac{2 - 3k}{(3k + 1)(3k - 2) + 2} c_{k-1} = \frac{2 - 3k}{9k^2 - 3k} c_{k-1} \]

\[ c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1} c_{1-1} = -\frac{1}{6} \]

\[ c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2} c_{2-1} = -\frac{4}{30} \left( -\frac{1}{6} \right) = \frac{1}{45} \]

\[ c_3 = \frac{2 - 3 \cdot 3}{9 \cdot 3^2 - 3 \cdot 3} c_{3-1} \]
Solve the Differential Equation

\[9x^2y'' + 3x^2y' + 2y = 0\]

\[r = \frac{1}{3}, \ c_0 = 1\]

\[c_k = \frac{3 - 3k - 3r}{9(k + r)(k + r - 1) + 2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3 \left(k + \frac{1}{3}\right) 3 \left(k + \frac{1}{3} - 1\right) + 2}c_{k-1}\]

\[= \frac{2 - 3k}{(3k + 1)(3k - 2) + 2}c_{k-1} = \frac{2 - 3k}{9k^2 - 3k}c_{k-1}\]

\[c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1}c_{1-1} = -\frac{1}{6}\]

\[c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2}c_{2-1} = -\frac{4}{30} \left(-\frac{1}{6}\right) = \frac{1}{45}\]

\[c_3 = \frac{2 - 3 \cdot 3}{9 \cdot 3^2 - 3 \cdot 3}c_{3-1} = -\frac{7}{72} \frac{1}{45}\]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3}, \; c_0 = 1 \]

\[
c_k = \frac{3 - 3k - 3r}{9(k+r)(k+r-1)+2}c_{k-1} = \frac{3 - 3k - 3 \cdot \frac{1}{3}}{3(k+\frac{1}{3})3(k+\frac{1}{3}-1)+2}c_{k-1}
\]

\[
= \frac{2 - 3k}{(3k+1)(3k-2)+2}c_{k-1} = \frac{2 - 3k}{9k^2 - 3k}c_{k-1}
\]

\[
c_1 = \frac{2 - 3 \cdot 1}{9 \cdot 1^2 - 3 \cdot 1}c_{1-1} = -\frac{1}{6}
\]

\[
c_2 = \frac{2 - 3 \cdot 2}{9 \cdot 2^2 - 3 \cdot 2}c_{2-1} = -\frac{4}{30} \left( -\frac{1}{6} \right) = \frac{1}{45}
\]

\[
c_3 = \frac{2 - 3 \cdot 3}{9 \cdot 3^2 - 3 \cdot 3}c_{3-1} = -\frac{7 \cdot 1}{72 \cdot 45} = -\frac{7}{3240}
\]
Solve the Differential Equation

$$9x^2y'' + 3x^2y' + 2y = 0$$

$$r = \frac{1}{3}$$

$$c_4 = \frac{2 - 3 \cdot 4}{9 \cdot 4^2 - 3 \cdot 4} c_4 - 1$$
Solve the Differential Equation
\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]
\[ r = \frac{1}{3} \]
\[ c_4 = \frac{2 - 3 \cdot 4}{9 \cdot 4^2 - 3 \cdot 4} c_{4-1} = -\frac{10}{132} \left( -\frac{7}{3240} \right) \]
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{1}{3} \]

\[ c_4 = \frac{2 - 3 \cdot 4}{9 \cdot 4^2 - 3 \cdot 4} c_{4-1} = -\frac{10}{132} \left( -\frac{7}{3240} \right) = \frac{7}{42768} \]
Solve the Differential Equation

\[ 9x^2 y'' + 3x^2 y' + 2y = 0 \]

\[ r = \frac{1}{3} \]

\[ c_4 = \frac{2 - 3 \cdot 4}{9 \cdot 4^2 - 3 \cdot 4} c_{4-1} = -\frac{10}{132} \left( -\frac{7}{3240} \right) = \frac{7}{42768} \]

\[ y_2 = x^{\frac{1}{3}} - \frac{1}{6} x^{\frac{1}{3}+1} + \frac{1}{45} x^{\frac{1}{3}+2} - \frac{7}{3240} x^{\frac{1}{3}+3} + \frac{7}{42768} x^{\frac{1}{3}+4} \]
Solve the Differential Equation

\[ 9x^2y'' + 3x^2y' + 2y = 0 \]

\[ r = \frac{1}{3} \]

\[ c_4 = \frac{2 - 3 \cdot 4}{9 \cdot 4^2 - 3 \cdot 4} c_{4-1} = -\frac{10}{132} \left( -\frac{7}{3240} \right) = \frac{7}{42768} \]

\[ y_2 = x^{\frac{1}{3}} - \frac{1}{6} x^{\frac{1}{3}+1} + \frac{1}{45} x^{\frac{1}{3}+2} - \frac{7}{3240} x^{\frac{1}{3}+3} + \frac{7}{42768} x^{\frac{1}{3}+4} \]

\[ y_2 = x^{\frac{1}{3}} - \frac{1}{6} x^{\frac{4}{3}} + \frac{1}{45} x^{\frac{7}{3}} - \frac{7}{3240} x^{\frac{10}{3}} + \frac{7}{42768} x^{\frac{13}{3}} \]
Checking Solutions When Only The First Few Terms are Available
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As for series solutions, because the “solution” is a truncated series, we cannot expect that the differential equation is exactly satisfied.
Checking Solutions When Only The First Few Terms are Available

As for series solutions, because the “solution” is a truncated series, we cannot expect that the differential equation is exactly satisfied.

\[
y_1(x) := x^3 - \frac{1}{6}x^3 + \frac{5}{252}x^3 - \frac{1}{567}x^3 + \frac{11}{88452}x^3
\]
Checking Solutions When Only The First Few Terms are Available

As for series solutions, because the “solution” is a truncated series, we cannot expect that the differential equation is exactly satisfied.

\[
y_1(x) := x^3 - \frac{1}{6} x^3 + \frac{5}{252} x^3 - \frac{1}{567} x^3 + \frac{11}{88452} x^3
\]

\[
9 \cdot x^2 \frac{d^2 y_1(x)}{dx^2} + 3 \cdot x \frac{d y_1(x)}{dx} + 2 \cdot y_1(x) \text{ collect, } x \rightarrow \frac{11}{6318} x^3
\]
Checking Solutions When Only The First Few Terms are Available

As for series solutions, because the “solution” is a truncated series, we cannot expect that the differential equation is exactly satisfied.

\[
y_1(x) := x^2 - \frac{1}{6} \cdot x^3 + \frac{5}{252} \cdot x^3 - \frac{1}{567} \cdot x^3 + \frac{11}{88452} x^3
\]

\[
y_2(x) := x^3 - \frac{1}{6} \cdot x^3 + \frac{1}{45} \cdot x^3 - \frac{7}{3240} \cdot x^3 + \frac{7}{42768} x^3
\]

\[
9 \cdot x^2 \cdot \frac{d^2}{dx^2} y_1(x) + 3 \cdot x \cdot \frac{d}{dx} y_1(x) + 2 \cdot y_1(x) \text{ collect, } x \to \frac{11}{6318} \cdot x^3
\]
Checking Solutions When Only The First Few Terms are Available

As for series solutions, because the “solution” is a truncated series, we cannot expect that the differential equation is exactly satisfied.

\[
y_1(x) := x^2 - \frac{1}{3}x^3 + \frac{5}{252}x^3 - \frac{1}{567}x^3 + \frac{11}{88452}x^3
\]

\[
9 \cdot x^2 \frac{d^2}{dx^2}y_1(x) + 3 \cdot x \frac{d}{dx}y_1(x) + 2 \cdot y_1(x) \text{ collect, } x \to \frac{17}{6318}x^3
\]

\[
y_2(x) := x^3 - \frac{1}{3}x^3 + \frac{1}{45}x^3 - \frac{7}{3240}x^3 + \frac{7}{42768}x^3
\]

\[
9 \cdot x^2 \frac{d^2}{dx^2}y_2(x) + 3 \cdot x \frac{d}{dx}y_2(x) + 2 \cdot y_2(x) \text{ collect, } x \to \frac{16}{42768}x^3
\]
Checking Solutions When Only The First Few Terms are Available

As before, for a correct approximation, low order terms should cancel.
Checking Solutions When Only The First Few Terms are Available

As before, for a correct approximation, low order terms should cancel.

We went to order $\frac{14}{3}$ for the first solution and to order $\frac{13}{3}$ for the second solution.
Checking Solutions When Only The First Few Terms are Available

As before, for a correct approximation, low order terms should cancel.

We went to order $\frac{14}{3}$ for the first solution and to order $\frac{13}{3}$ for the second solution.
And, of course, we use a computer algebra system.