The Differential Equation for a Vibrating String

Bernd Schröder

Modeling Assumptions

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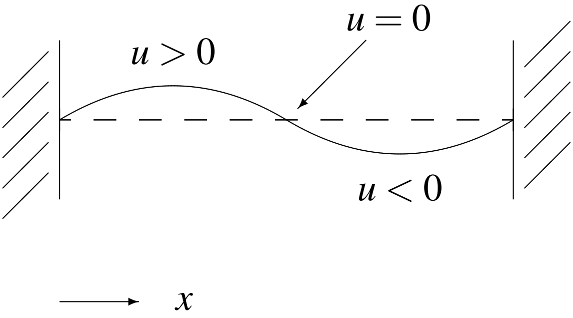

Decomposing the Tensile Force

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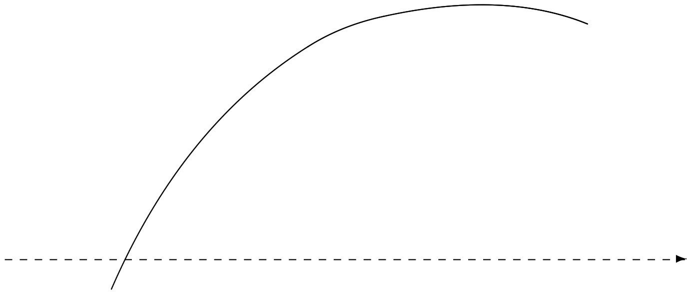

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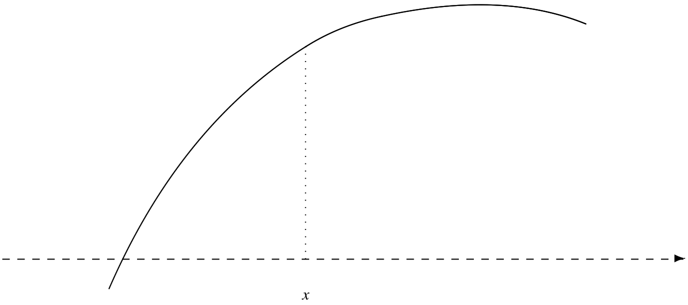

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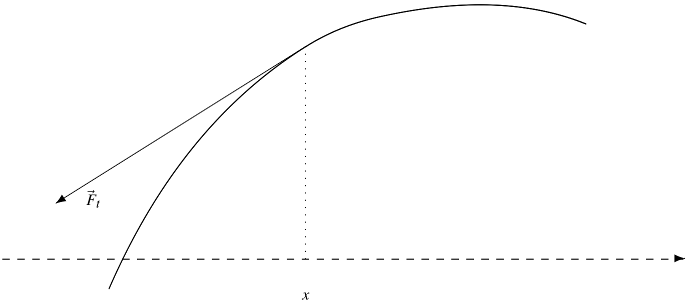

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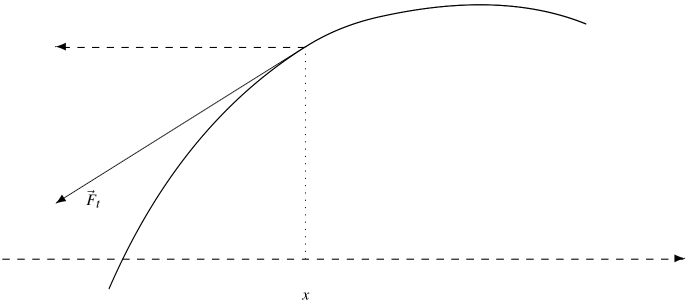

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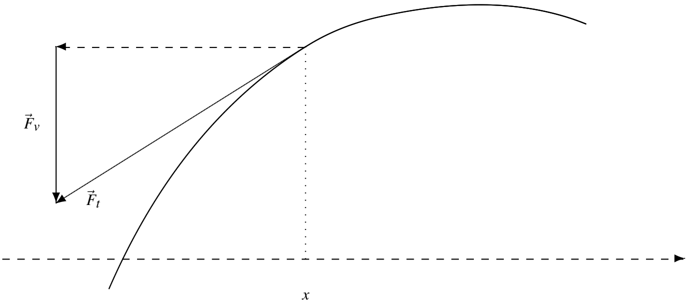

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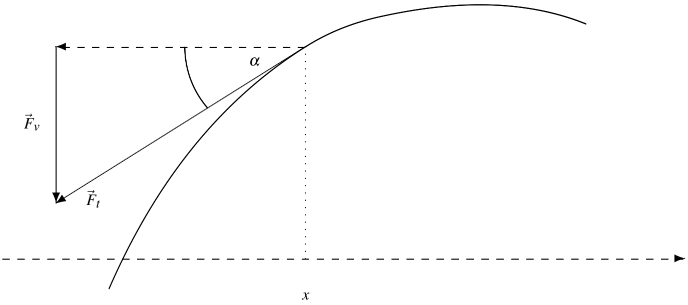

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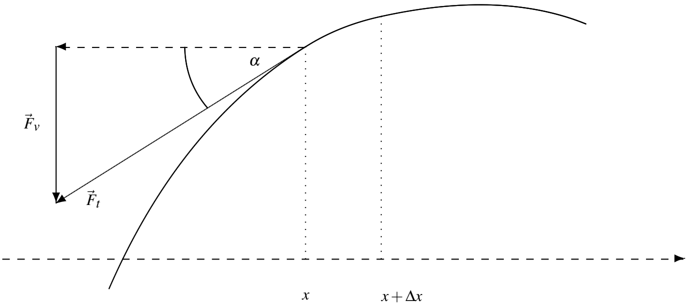

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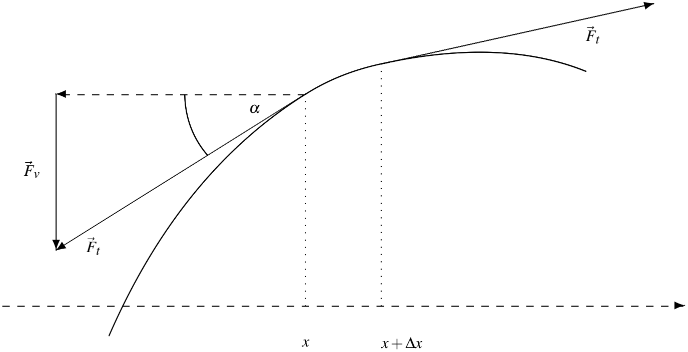

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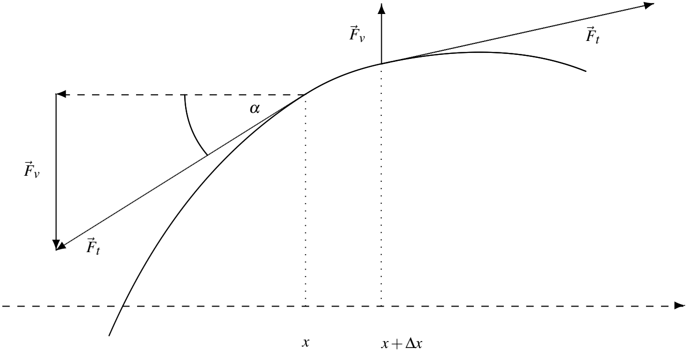

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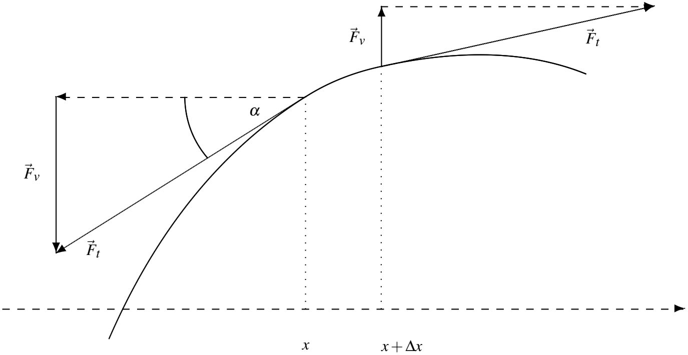

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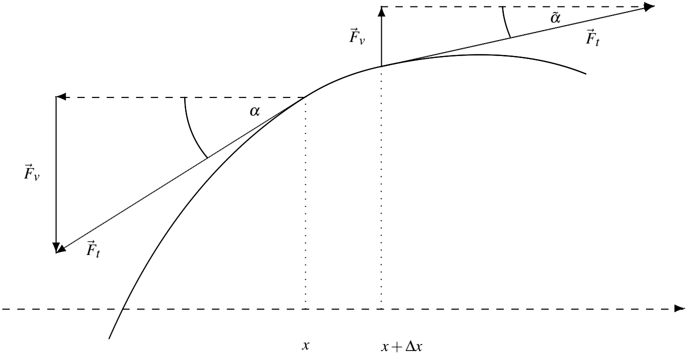

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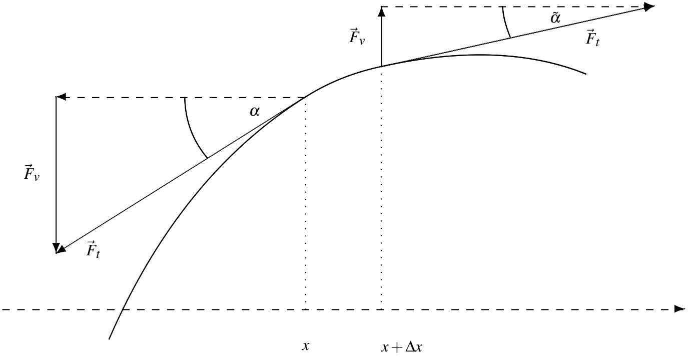

$$F(x) \approx F_v(x+\Delta x) - F_v(x)$$

The Vertical Force at a Point

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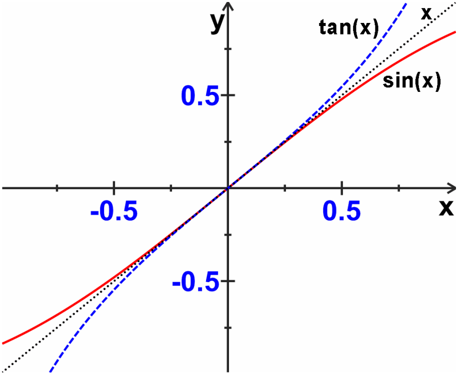

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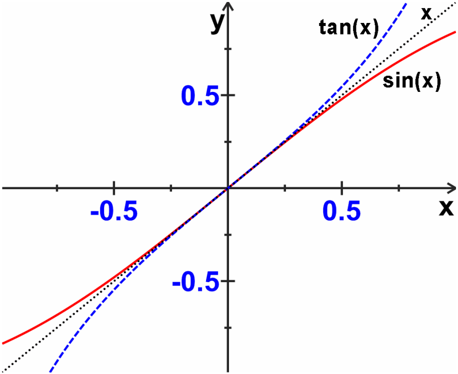

$0.25 \approx 14.3^\circ$

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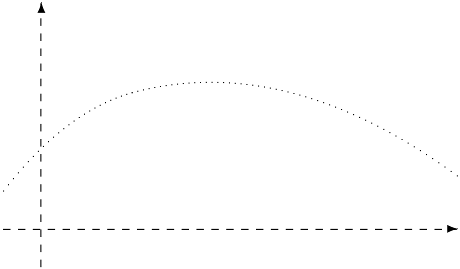

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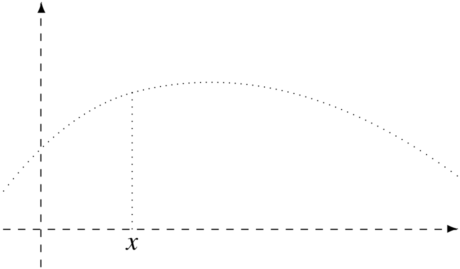

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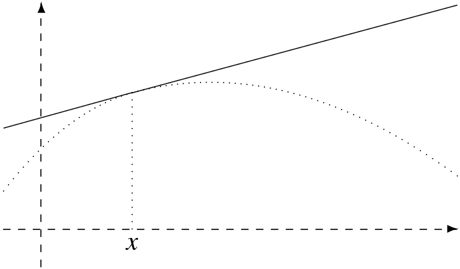

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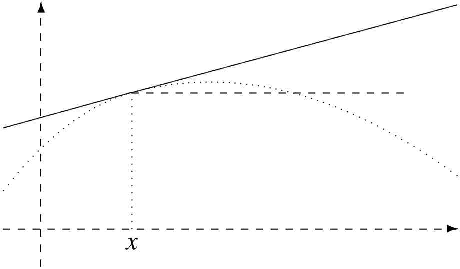

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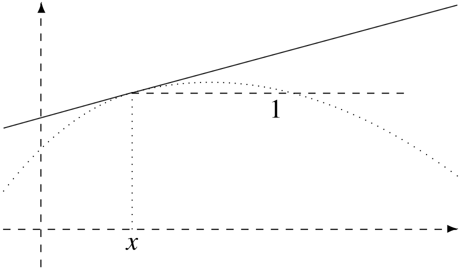

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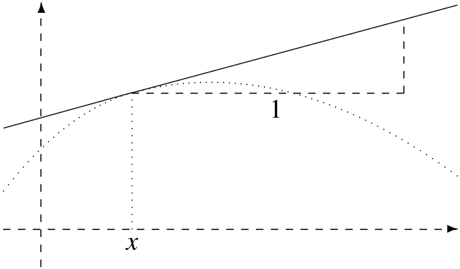

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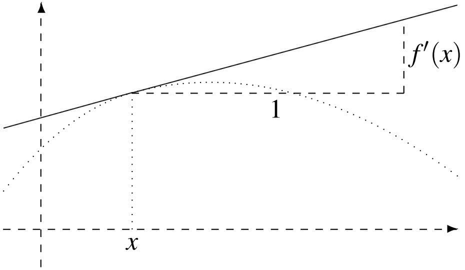

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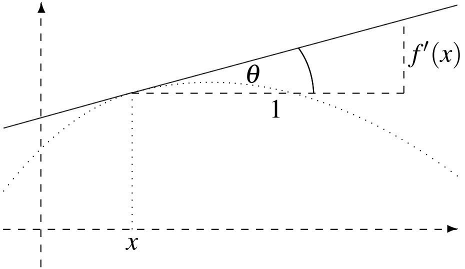

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Using Newton's Second Law

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$$ma$$

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The One-Dimensional Wave Equation

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The equation of motion for small oscillations of a frictionless string is

$$\frac{\partial^2}{\partial x^2}u(x,t) = k\frac{\partial^2}{\partial t^2}u(x,t),$$

where $k = \frac{\rho_l}{F_t} > 0$, with ρ_l being the linear density of the string and F_t being the tensile force.

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The cancellation of the Δx was "clean".