The Differential Equation for a Vibrating String

Bernd Schröder
Modeling Assumptions

1. The string is made up of individual particles that move vertically.
2. \( u(x, t) \) is the vertical displacement from equilibrium of the particle at horizontal position \( x \) and at time \( t \).
   - \( u > 0 \)
   - \( u < 0 \)
   - \( u = 0 \)
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\[ u > 0 \quad u = 0 \quad u < 0 \]
Decomposing the Tensile Force
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The Differential Equation for a Vibrating String
Decomposing the Tensile Force

\[ \vec{F}_t \]

\[ x \]
Decomposing the Tensile Force

\( \vec{F}_t \)
Decomposing the Tensile Force

\[ \vec{F}_v \]
\[ \vec{F}_t \]

\[ x \]
Decomposing the Tensile Force

\[
\vec{F}_v + \alpha \vec{F}_t
\]
Decomposing the Tensile Force

\[ \vec{F}_v \]

\[ \vec{F}_t \]

\[ \alpha \]

\[ x \]

\[ x + \Delta x \]

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The Differential Equation for a Vibrating String
Decomposing the Tensile Force

\[
\vec{F}_v + \vec{F}_t = \vec{F}_t
\]

\[
\alpha \:\vec{F}_v + \vec{F}_t = \vec{F}_t
\]

\[
\alpha \vec{F}_v = \vec{0}
\]

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Decomposing the Tensile Force

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Decomposing the Tensile Force

\[ \vec{F}_v \] 
\[ \vec{F}_t \] 
\[ x \]
\[ x + \Delta x \]

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The Differential Equation for a Vibrating String
Decomposing the Tensile Force

\[ \vec{F}_v(x) \approx \vec{F}_v(x + \Delta x) - \vec{F}_v(x) \]

\[ F(x) = F_v(x + \Delta x) - F_v(x) \]

\[ \vec{F}_v \]

\[ \vec{F}_t \]

\[ \alpha \]

\[ \alpha \hat{\alpha} \]

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The Differential Equation for a Vibrating String
Decomposing the Tensile Force

\[ F(x) \approx F_v(x + \Delta x) - F_v(x) \]
The Vertical Force at a Point

\[ F(x) \approx F_v(x + \Delta x) - F_v(x) = F_t \sin(\tilde{\alpha}) - F_t \sin(\alpha) \]
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\[ 0.25 \approx 14.3^\circ \]
The Vertical Force at a Point

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## The Vertical Force at a Point

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\approx F_t \tan(\tilde{\alpha}) - F_t \tan(\alpha) \quad (\tan(\theta) = f'(x))
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\[ = F_t \left( \frac{d}{dx} u(x + \Delta x) - \frac{d}{dx} u(x) \right) \]
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\[ = F_t \left( \frac{d}{dx} u(x + \Delta x) - \frac{d}{dx} u(x) \right) \quad (f(x + \Delta x) \approx f(x) + f'(x) \Delta x) \]
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\[ \approx F_t \left( \frac{d}{dx} u(x) + \Delta x \cdot \frac{d^2}{dx^2} u(x) \right) \]
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\[ = F_t \Delta x \frac{d^2}{dx^2} u(x) \]
Using Newton’s Second Law

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Using Newton’s Second Law

\[ ma \]
Using Newton’s Second Law

\[ ma = F(x) \]
Using Newton’s Second Law

\[ ma = F(x) = F_t \Delta x \frac{\partial^2}{\partial x^2} u(x, t) \]
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\[ ma = F(x) = F_t \Delta x \frac{\partial^2}{\partial x^2} u(x, t) \]

\[ \rho_l \Delta x \]
Using Newton’s Second Law

\[ ma = F(x) = F_{t\Delta x} \frac{\partial^2}{\partial x^2} u(x, t) \]

\[ \rho_{l\Delta x} \frac{\partial^2}{\partial t^2} u(x, t) \]
Using Newton’s Second Law

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Using Newton’s Second Law

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\[ \rho_l \Delta x \frac{\partial^2}{\partial t^2} u(x, t) = F_t \Delta x \frac{\partial^2}{\partial x^2} u(x, t) \]

\[ \frac{\rho_l}{F_t} \frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) \]
The One-Dimensional Wave Equation

The equation of motion for small oscillations of a frictionless string is

$$\frac{\partial^2}{\partial x^2} u(x, t) = k \frac{\partial^2}{\partial t^2} u(x, t),$$

where $k = \frac{\rho l F_t}{\rho l} > 0$, with $\rho l$ being the linear density of the string and $F_t$ being the tensile force. This equation is also called the one-dimensional wave equation. Our derivation is valid for small oscillations and negligible friction.

The cancellation of the $\Delta x$ was "clean".
The One-Dimensional Wave Equation

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