Power Series (Overview)

Bernd Schröder

Facts About Power Series

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2. There is a nonnegative R (the **radius of convergence**), which could be infinity, so that the power series converges for every x in (x_0-R, x_0+R) and so that the power series diverges for every x not in (x_0-R, x_0+R).

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3. For $R<\infty$, to decide about convergence at x_0-R and x_0+R, further convergence tests for series are needed.
4. Standard way to compute the radius of convergence for many series: Apply the **ratio test**. The power series converges for all x for which $\lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_nx^n}\right|<1$ and it diverges for all x for which the limit is greater than 1.

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which is always true. Hence $R = \infty$.

Taylor Series

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4. Taylor series about $x_0 = 0$ ("the usual expansion point") are also called McLaurin series.
5. The Taylor series of e^x about $x_0 = 0$ is $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$.

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