Reduction of Order

Bernd Schröder

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4. There are two ways to proceed.
   4.1 We can substitute $y = uy_1$ into the equation. This leads to a first order differential equation, which explains the name.
   4.2 Or, we can use the formula $y_2 = y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$, which is obtained by doing the above substitution symbolically.

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v' &= -\frac{4v}{x}
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y_2 &= y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx \\
y_2 &= x^2 \int \frac{e^{-\int 0\,dx}}{\left(x^2\right)^2}\,dx
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y_2 &= -\frac{1}{3}x^{-1} \\
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- The formula is quicker, but it only works for equations of the form $y'' + P(x)y' + Q(x)y = 0$.
- The setup $y_2 = uy_1$ requires more computation, but it is easier to remember, and it also works for some equations that are not of the form $y'' + P(x)y' + Q(x)y = 0$.