Reduction of Order

Bernd Schröder
What is Reduction of Order?

Typically, reduction of order is applied to second order linear differential equations of the form:

\[ y'' + P(x)y' + Q(x)y = 0. \]

We must already have one solution \( y_1 \) of the equation.

Reduction of order assumes there is a second, linearly independent solution of the form \( y = uy_1 \).

There are two ways to proceed.

1. We can substitute \( y = uy_1 \) into the equation. This leads to a first order differential equation, which explains the name.
2. Or, we can use the formula \( y_2 = y_1 \int e^{-\int P \, dx} y_1^2 \, dx \), which is obtained by doing the above substitution symbolically.
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2. We must already have one solution $y_1$ of the equation.

3. Reduction of order assumes there is a second, linearly independent solution of a the form $y = uy_1$.

4. There are two ways to proceed.

   4.1 We can substitute $y = uy_1$ into the equation. This leads to a first order differential equation, which explains the name.

   4.2 Or, we can use the formula $y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx$, which is obtained by doing the above substitution symbolically.
Use Reduction of Order to Find a Second Solution for \( x^2y'' - 2y = 0 \), given that \( y_1 = x^2 \).
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$$y = ux^2$$
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\[ y = ux^2 \]
\[ y' = u'x^2 + u2x \]
Use Reduction of Order to Find a Second Solution for $x^2 y'' - 2y = 0$, given that $y_1 = x^2$.

\[ y = ux^2 \]
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\[ y'' = u''x^2 + u'4x + u2 \]
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$$y = ux^2$$
$$y' = u'x^2 + u2x$$
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$$x^2 \left( u''x^2 + u'4x + u2 \right) - 2ux^2 = 0$$
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$$y' = u' x^2 + u2x$$
$$y'' = u'' x^2 + u'4x + u2$$

$$x^2 \left( u'' x^2 + u'4x + u2 \right) - 2ux^2 = 0$$
$$u'' x^4 + u'4x^3 + u2x^2 - 2ux^2 = 0$$
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\begin{align*}
y &= ux^2 \\
 y' &= u'x^2 + u2x \\
 y'' &= u''x^2 + u'4x + u2
\end{align*}
\]

\[
x^2 \left( u''x^2 + u'4x + u2 \right) - 2ux^2 = 0
\]

\[
\begin{align*}
u''x^4 + u'4x^3 + u2x^2 - 2ux^2 &= 0 \\
u''x^4 + u'4x^3 &= 0
\end{align*}
\]
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

$$y = ux^2$$
$$y' = u'x^2 + u2x$$
$$y'' = u''x^2 + u'4x + u2$$

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$$u''x^4 + u'4x^3 = 0$$

$$u''x + u'4 = 0$$
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\begin{align*}
y &= ux^2 \\
y' &= u'x^2 + u2x \\
y'' &= u''x^2 + u'4x + u2 \\
x^2 \left( u''x^2 + u'4x + u2 \right) - 2ux^2 &= 0 \\
u''x^4 + u'4x^3 + u2x^2 - 2ux^2 &= 0 \\
u''x^4 + u'4x^3 &= 0 \\
u''x + u'4 &= 0 \\
v &:= u'
\end{align*}
\]
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

Let $y = ux^2$

Then,

$y' = u'x^2 + u2x$

$y'' = u''x^2 + u'4x + u2$

$x^2\left(u''x^2 + u'4x + u2\right) - 2ux^2 = 0$

$u''x^4 + u'4x^3 + u2x^2 - 2ux^2 = 0$

$u''x^4 + u'4x^3 = 0$

$u''x + u'4 = 0$

$v := u'$

$v' = -\frac{4v}{x}$
Use Reduction of Order to Find a Second Solution for \( x^2y'' - 2y = 0 \), given that \( y_1 = x^2 \).

\[
\frac{dv}{dx} = -\frac{4v}{x}
\]
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

\[
\begin{align*}
\frac{dv}{dx} &= -\frac{4v}{x} \\
\frac{dv}{v} &= -4 \frac{dx}{x}
\end{align*}
\]
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

\[
\frac{dv}{dx} = -\frac{4v}{x} \\
\int \frac{dv}{v} = \int -\frac{4 \, dx}{x}
\]
Use Reduction of Order to Find a Second Solution for $x^2y''' - 2y = 0$, given that $y_1 = x^2$.

$$\frac{dv}{dx} = -\frac{4v}{x}$$

$$\int \frac{dv}{v} = \int -\frac{4}{x} dx$$

$$\ln(v) = -4\ln(x) + c$$
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

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\begin{align*}
\frac{dv}{dx} &= -\frac{4v}{x} \\
\int \frac{dv}{v} &= \int -4 \frac{dx}{x} \\
\ln(v) &= -4 \ln(x)
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Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

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\frac{dv}{dx} = -\frac{4v}{x}
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\[
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\ln(v) = -4 \ln(x)
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\[
v = e^{-4 \ln(x)}
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Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

$$\frac{dv}{dx} = -\frac{4v}{x}$$

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$$v = e^{-4 \ln(x)} = x^{-4}$$
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

\[
\frac{dv}{dx} = -\frac{4v}{x}
\]

\[
\int \frac{dv}{v} = \int -\frac{4}{x} \, dx
\]

\[
\ln(v) = -4 \ln(x)
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v = e^{-4 \ln(x)} = x^{-4}
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$u$
Use Reduction of Order to Find a Second Solution for \( x^2y'' - 2y = 0 \), given that \( y_1 = x^2 \).

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\frac{dv}{dx} = -\frac{4v}{x}, \\
\int \frac{dv}{v} = \int -4 \frac{dx}{x}, \\
\ln(v) = -4 \ln(x), \\
v = e^{-4 \ln(x)} = x^{-4}
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u = \int v \, dx
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Use Reduction of Order to Find a Second Solution for \(x^2y'' - 2y = 0\), given that \(y_1 = x^2\).

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\begin{align*}
\frac{dv}{dx} &= -\frac{4v}{x} \\
\int \frac{dv}{v} &= \int -\frac{4}{x} dx \\
\ln(v) &= -4\ln(x) \\
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\[
u = \int v \, dx = \int x^{-4} \, dx = -\frac{1}{3}x^{-3}
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Use Reduction of Order to Find a Second Solution for \( x^2y'' - 2y = 0 \), given that \( y_1 = x^2 \).

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y = x^2 u
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\int \frac{dv}{v} = \int -\frac{4}{x} dx \\
\ln(v) = -4 \ln(x) \\
v = e^{-4 \ln(x)} = x^{-4} \\
u = \int v \, dx = \int x^{-4} \, dx = -\frac{1}{3} x^{-3} \\
y = x^2u = x^2 \left(-\frac{1}{3} x^{-3}\right)
\]
Use Reduction of Order to Find a Second Solution for $x^2y'' - 2y = 0$, given that $y_1 = x^2$.

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\begin{align*}
\frac{dv}{dx} &= -\frac{4v}{x} \\
\int \frac{dv}{v} &= \int -\frac{4}{x} \, dx \\
\ln(v) &= -4 \ln(x) \\
v &= e^{-4 \ln(x)} = x^{-4} \\
u &= \int v \, dx = \int x^{-4} \, dx = -\frac{1}{3}x^{-3} \\
y &= x^2u = x^2 \left( -\frac{1}{3}x^{-3} \right) = -\frac{1}{3}x^{-1}
\end{align*}
\]
Use the Reduction of Order Formula and $y_1 = x^2$
to Find a Second Solution for $x^2y'' - 2y = 0$. 

$$y_2 = y_1 \int e^{-\int P \, dx} \left( x^2 \right)^2 \, dx = x^2 \int 1 \, x^4 \, dx = x^2(\frac{-1}{3}x^3) = -\frac{1}{3}x^5.$$
Use the Reduction of Order Formula and $y_1 = x^2$ to Find a Second Solution for $x^2y'' - 2y = 0$.

$$y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx$$
Use the Reduction of Order Formula and $y_1 = x^2$ to Find a Second Solution for $x^2y'' - 2y = 0$.

\[
y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx
\]

\[
y_2 = x^2 \int \frac{e^{-\int 0 \, dx}}{(x^2)^2} \, dx
\]

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Louisiana Tech University, College of Engineering and Science
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Use the Reduction of Order Formula and $y_1 = x^2$ to Find a Second Solution for $x^2y'' - 2y = 0$. 

\[ y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx \]

\[ y_2 = x^2 \int \frac{e^{-\int 0 \, dx}}{(x^2)^2} \, dx \]

\[ = x^2 \int \frac{1}{x^4} \, dx \]
Use the Reduction of Order Formula and $y_1 = x^2$ to Find a Second Solution for $x^2y'' - 2y = 0$.

$$y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx$$

$$y_2 = x^2 \int \frac{e^{-\int 0 \, dx}}{(x^2)^2} \, dx$$

$$= x^2 \int \frac{1}{x^4} \, dx$$

$$= x^2 \left( -\frac{1}{3} \frac{1}{x^3} \right)$$
Use the Reduction of Order Formula and $y_1 = x^2$ to Find a Second Solution for $x^2y'' - 2y = 0$.

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y_2 = y_1 \int \frac{e^{-\int P \, dx}}{y_1^2} \, dx
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y_2 = x^2 \int \frac{e^{-\int 0 \, dx}}{(x^2)^2} \, dx
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\[
= x^2 \int \frac{1}{x^4} \, dx
\]

\[
= x^2 \left( -\frac{1}{3} x^3 \right)
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\[
= -\frac{1}{3} x^{-1}
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Does $y_2 = -\frac{1}{3}x^{-1}$ Really Solve $x^2y'' - 2y = 0$?
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\[y_2' = \frac{1}{3}x^{-2}\]
Does $y_2 = -\frac{1}{3}x^{-1}$ Really Solve $x^2y'' - 2y = 0$?

\[
\begin{align*}
y_2 &= -\frac{1}{3}x^{-1} \\
y_2' &= \frac{1}{3}x^{-2} \\
y_2'' &= \frac{1}{3}(-2)x^{-3}
\end{align*}
\]
Does \( y_2 = -\frac{1}{3} x^{-1} \) Really Solve \( x^2 y'' - 2y = 0 \)?

\[
\begin{align*}
  y_2 & = -\frac{1}{3} x^{-1} \\
  y'_2 & = \frac{1}{3} x^{-2} \\
  x^2 y''_2 & = x^2 \frac{1}{3} (-2) x^{-3}
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\[
\begin{align*}
y_2 &= -\frac{1}{3} x^{-1} \\
y_2' &= \frac{1}{3} x^{-2} \\
x^2 y_2'' &= x^2 \frac{1}{3} (-2) x^{-3} \\
      &= 2 \left( -\frac{1}{3} x^{-1} \right)
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\[
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\]
\[
x^2y'' = x^2 \frac{1}{3}(-2)x^{-3}
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= 2 \left(-\frac{1}{3}x^{-1}\right)
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= 2 \left(-\frac{1}{3}x^{-1}\right) \\
= 2y_2 \quad \checkmark
\]
Comparing the Two Approaches

The formula is quicker, but it only works for equations of the form $y'' + P(x)y' + Q(x)y = 0$.

The setup $y_2 = uy_1$ requires more computation, but it is easier to remember, and it also works for some equations that are not of the form $y'' + P(x)y' + Q(x)y = 0$.
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