An Initial Value Problem for a Separable Differential Equation

Bernd Schröder

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That's it.

Solve the Initial Value Problem

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y' &= x\sin(x)\left(y + \frac{1}{y}\right) \\
\frac{dy}{dx} &= x\sin(x)\left(\frac{y^2+1}{y}\right) \\
\frac{y}{y^2+1}\,dy &= x\sin(x)\,dx
\end{aligned}$$

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$$\begin{aligned} \int \frac{y}{y^2+1}\,dy &= \int \frac{y}{u}\,\frac{du}{2y} \\ &= \frac{1}{2}\int \frac{1}{u}\,du \end{aligned}$$

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$$\begin{aligned}\int x\sin(x)\,dx &= x\big(-\cos(x)\big) - \int 1\cdot\big(-\cos(x)\big)\,dx \\ &= -x\cos(x) + \int \cos(x)\,dx\end{aligned}$$

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... Continuing Where We Left Off.

$$\int \frac{y}{y^2+1}\,dy \quad = \quad \int x\sin(x)\,dx$$

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$$
\begin{aligned}
\int \frac{y}{y^2+1}\,dy &= \int x\sin(x)\,dx \\
\frac{1}{2}\ln\left|y^2+1\right| &=
\end{aligned}
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$$\int \frac{y}{y^2+1}\,dy = \int x\sin(x)\,dx$$

$$\frac{1}{2}\ln\left|y^2+1\right| = \sin(x) - x\cos(x) + c$$

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\begin{aligned}
\int \frac{y}{y^2+1}\,dy &= \int x\sin(x)\,dx \\
\frac{1}{2}\ln\left|y^2+1\right| &= \sin(x)-x\cos(x)+c \\
\ln\left|y^2+1\right| &= 2\sin(x)-2x\cos(x)+c
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y^2+1 &= e^{2\sin(x)-2x\cos(x)+c}
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y^2+1 &= ke^{2\sin(x)-2x\cos(x)} \\
y^2 &= ke^{2\sin(x)-2x\cos(x)} - 1 \\
y &= \pm\sqrt{ke^{2\sin(x)-2x\cos(x)} - 1}
\end{aligned}$$

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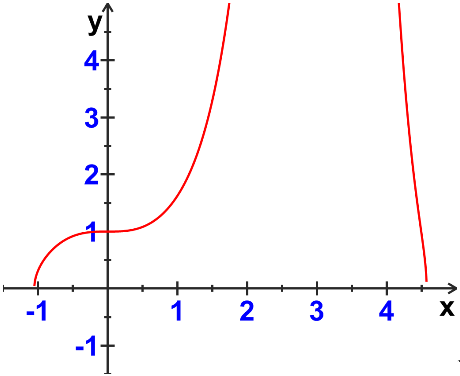
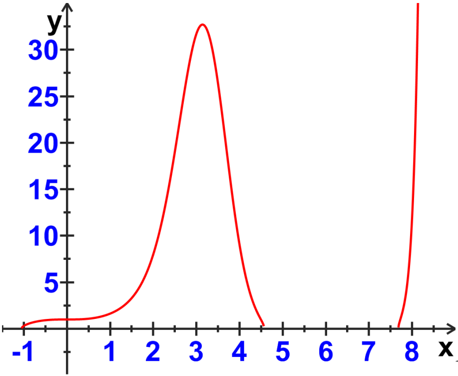


$$y(x) = \sqrt{2e^{2\sin(x) - 2x\cos(x)} - 1}$$

Solving the Initial Value Problem



$$y(x) = \sqrt{2e^{2\sin(x)-2x\cos(x)} - 1}$$

Does $y(x)=\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}$ Solve the IVP $y'=x\sin(x)y+\frac{x\sin(x)}{y}, y(0)=1$?

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$$\frac{d}{dx}\left(\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}\right)$$

$$= \frac{2e^{2\sin(x)-2x\cos(x)}}{2\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}$$

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$$=\frac{2e^{2\sin(x)-2x\cos(x)}}{2\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}\big(2\cos(x)-2\cos(x)+2x\sin(x)\big)$$

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$$\begin{aligned}
&\frac{d}{dx}\left(\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}\right)\\
&= \frac{2e^{2\sin(x)-2x\cos(x)}}{2\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}\big(2\cos(x)-2\cos(x)+2x\sin(x)\big)\\
&= \frac{2e^{2\sin(x)-2x\cos(x)}}{2\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}2x\sin(x)\\
&= \frac{2e^{2\sin(x)-2x\cos(x)}}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}x\sin(x)
\end{aligned}$$

$$x\sin(x)y + \frac{x\sin(x)}{y}$$

$$x\sin(x)y+\frac{x\sin(x)}{y}$$

$$= x\sin(x)\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}+\frac{x\sin(x)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}$$

$$
\begin{aligned}
& x\sin(x)y + \frac{x\sin(x)}{y} \\
= \quad & x\sin(x)\sqrt{2e^{2\sin(x)-2x\cos(x)}-1} + \frac{x\sin(x)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}} \\
= \quad & \frac{x\sin(x)\left(\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}\right)^2 + x\sin(x)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}
\end{aligned}
$$

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= \quad & \frac{x\sin(x)\left(2e^{2\sin(x)-2x\cos(x)}-1+1\right)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}
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= \quad & \frac{x\sin(x)\left(2e^{2\sin(x)-2x\cos(x)}-1+1\right)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}} \\
= \quad & x\sin(x)\frac{2e^{2\sin(x)-2x\cos(x)}}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}}
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= \quad & \frac{x\sin(x)\left(2e^{2\sin(x)-2x\cos(x)}-1+1\right)}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}} \\
= \quad & x\sin(x)\frac{2e^{2\sin(x)-2x\cos(x)}}{\sqrt{2e^{2\sin(x)-2x\cos(x)}-1}} \qquad \surd
\end{aligned}
$$