Separable Differential Equations

Bernd Schröder

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2. That is, a differential equation is separable if the terms that are not equal to y' can be factored into a factor that only depends on x and another factor that only depends on y.
3. The solution method for separable differential equations looks like regular algebra with the added caveat that we use integrals to undo the differentials dx and dy from $y' = \frac{dy}{dx}$.

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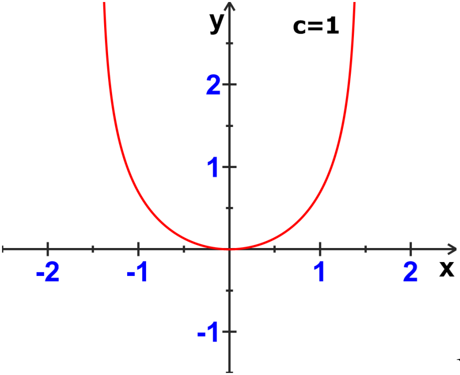

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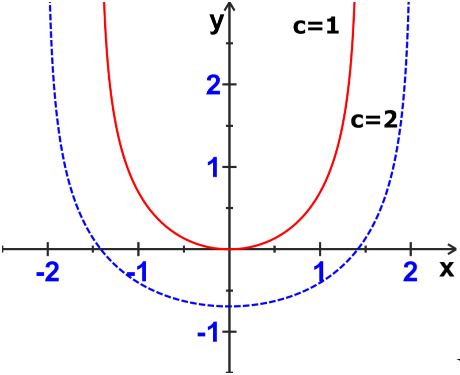

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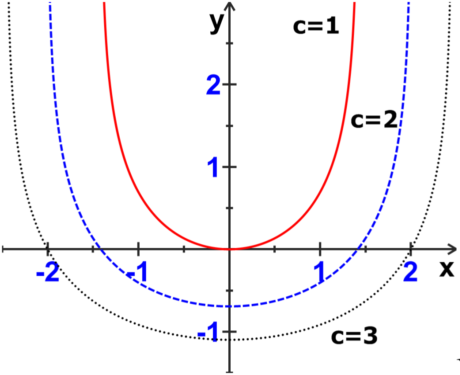

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  We don't.
- So we should always check the result.

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x\frac{1}{C - \frac{1}{2}x^2} &= x\frac{1}{C - \frac{1}{2}x^2} \qquad \checkmark
\end{aligned}$$