Separation of Variables – Bessel Equations

Bernd Schröder

Separation of Variables

1. Solution technique for partial differential equations.

Separation of Variables

1. Solution technique for partial differential equations.
2. If the unknown function u depends on variables r, θ, t, we assume there is a solution of the form $u = R(r)D(\theta)T(t)$.

Separation of Variables

1. Solution technique for partial differential equations.
2. If the unknown function u depends on variables r, θ, t, we assume there is a solution of the form $u = R(r)D(\theta)T(t)$.
3. The special form of this solution function allows us to replace the original partial differential equation with several ordinary differential equations.

Separation of Variables

1. Solution technique for partial differential equations.
2. If the unknown function u depends on variables r, θ, t, we assume there is a solution of the form $u = R(r)D(\theta)T(t)$.
3. The special form of this solution function allows us to replace the original partial differential equation with several ordinary differential equations.
4. Key step: If $f(t) = g(r, \theta)$, then f and g must be constant.

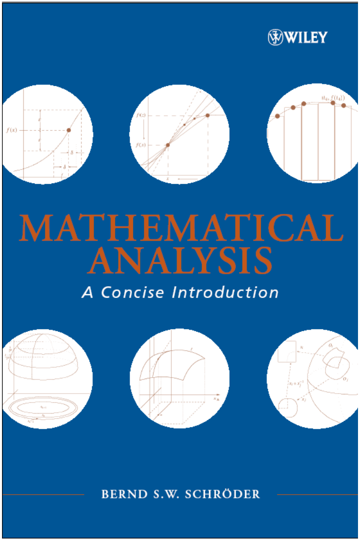
Separation of Variables

1. Solution technique for partial differential equations.
2. If the unknown function u depends on variables r, θ, t, we assume there is a solution of the form $u = R(r)D(\theta)T(t)$.
3. The special form of this solution function allows us to replace the original partial differential equation with several ordinary differential equations.
4. Key step: If $f(t) = g(r, \theta)$, then f and g must be constant.
5. Solutions of the ordinary differential equations we obtain must typically be processed some more to give useful results for the partial differential equations.

Separation of Variables

1. Solution technique for partial differential equations.
2. If the unknown function u depends on variables r, θ, t, we assume there is a solution of the form $u = R(r)D(\theta)T(t)$.
3. The special form of this solution function allows us to replace the original partial differential equation with several ordinary differential equations.
4. Key step: If $f(t) = g(r, \theta)$, then f and g must be constant.
5. Solutions of the ordinary differential equations we obtain must typically be processed some more to give useful results for the partial differential equations.
6. Some very powerful and deep theorems can be used to formally justify the approach for many equations involving the Laplace operator.

How Deep?



How Deep?

plus about 200 pages of really awesome functional analysis.

The Equation $\Delta u = k\frac{\partial u}{\partial t}$

The Equation $\Delta u = k\frac{\partial u}{\partial t}$

1. It's the heat equation.

The Equation $\Delta u = k\frac{\partial u}{\partial t}$

1. It's the heat equation.
2. Consideration in two dimensions may mean we analyze heat transfer in a thin sheet of metal.

The Equation $\Delta u = k\frac{\partial u}{\partial t}$

1. It's the heat equation.
2. Consideration in two dimensions may mean we analyze heat transfer in a thin sheet of metal.
3. It may also mean that we are working with a cylindrical geometry in which there is no variation in the z-direction. (Heating a metal cylinder in a water bath.)

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t)$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t)$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT + \frac{1}{r}R'DT$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= k\frac{\partial u}{\partial t} \\
u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T & \\
\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT &= k\frac{\partial}{\partial t}RDT \\
R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T &= kRDT'
\end{aligned}
$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T = kRDT'$$

$$\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= k\frac{\partial u}{\partial t} \\
u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T & \\
\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT &= k\frac{\partial}{\partial t}RDT \\
R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T &= kRDT'
\end{aligned}$$

$$\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T = kRDT'$$

$$\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T} = -\lambda^2$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= k\frac{\partial u}{\partial t} \\
u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T && \\
\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT &= k\frac{\partial}{\partial t}RDT \\
R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T &= kRDT' \\
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T} &= -\lambda^2
\end{aligned}$$

Constant is negative,

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k\frac{\partial u}{\partial t}$$

$$u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T$$

$$\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT = k\frac{\partial}{\partial t}RDT$$

$$R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T = kRDT'$$

$$\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T} = -\lambda^2$$

Constant is negative, because $\frac{T'}{T} = \frac{c}{k}$ gives $T = ae^{\frac{c}{k}t}$.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= k\frac{\partial u}{\partial t} \\
u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T \\
\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT &= k\frac{\partial}{\partial t}RDT \\
R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T &= kRDT' \\
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T} &= -\lambda^2
\end{aligned}$$

Constant is negative, because $\frac{T'}{T} = \frac{c}{k}$ gives $T = ae^{\frac{c}{k}t}$. Now $k > 0$ forces $c < 0$,

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Temporal Part)

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= k\frac{\partial u}{\partial t} \\
u(r,\theta,t) = R(r)D(\Theta)T(t) = R\cdot D\cdot T & \\
\frac{\partial^2}{\partial r^2}RDT + \frac{1}{r}\frac{\partial}{\partial r}RDT + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}RDT &= k\frac{\partial}{\partial t}RDT \\
R''DT + \frac{1}{r}R'DT + \frac{1}{r^2}RD''T &= kRDT' \\
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = k\frac{T'}{T} &= -\lambda^2
\end{aligned}$$

Constant is negative, because $\frac{T'}{T} = \frac{c}{k}$ gives $T = ae^{\frac{c}{k}t}$. Now $k > 0$ forces $c < 0$, otherwise temperature would increase exponentially with no energy input.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} = -\lambda^2$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD}
\end{aligned}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D}
\end{aligned}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$
\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}
$$

The constant is nonnegative:

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D+\frac{1}{r}R'D+\frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D+\frac{1}{r}R'D}{RD}+\lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R''+rR'}{R}+r^2\lambda^2 &= -\frac{D''}{D}=\nu^2
\end{aligned}$$

The constant is nonnegative: $-\frac{D''}{D}=c$ leads to $D''+cD=0$.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}$$

The constant is nonnegative: $-\frac{D''}{D} = c$ leads to $D'' + cD = 0$.
But D must be 2π-periodic.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}$$

The constant is nonnegative: $-\frac{D''}{D} = c$ leads to $D'' + cD = 0$. But D must be 2π-periodic. For negative c we get nonperiodic exponential solutions.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}$$

The constant is nonnegative: $-\frac{D''}{D} = c$ leads to $D'' + cD = 0$. But D must be 2π-periodic. For negative c we get nonperiodic exponential solutions. Thus $c = \nu^2$, where ν is a nonnegative integer,

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Azimuthal Part)

$$\begin{aligned}
\frac{R''D + \frac{1}{r}R'D + \frac{1}{r^2}RD''}{RD} &= -\lambda^2 \\
\frac{R''D + \frac{1}{r}R'D}{RD} + \lambda^2 &= -\frac{\frac{1}{r^2}RD''}{RD} \\
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= -\frac{D''}{D} = \nu^2
\end{aligned}$$

The constant is nonnegative: $-\frac{D''}{D} = c$ leads to $D'' + cD = 0$. But D must be 2π-periodic. For negative c we get nonperiodic exponential solutions. Thus $c = \nu^2$, where ν is a nonnegative integer, because then $D(\theta) = c_1\cos(\nu\theta) + c_2\sin(\nu\theta)$, which is 2π-periodic.

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Radial Part)

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Radial Part)

$$\frac{r^2R'' + rR'}{R} + r^2\lambda^2 = \nu^2$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Radial Part)

$$\begin{aligned} \frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= \nu^2 \\ r^2R'' + rR' + r^2\lambda^2 R &= \nu^2 R \end{aligned}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Radial Part)

$$\begin{aligned}
\frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= \nu^2 \\
r^2R'' + rR' + r^2\lambda^2 R &= \nu^2 R \\
r^2R'' + rR' + \left(\lambda^2 r^2 - \nu^2\right) R &= 0
\end{aligned}$$

Separating the Equation $\Delta u = k\frac{\partial u}{\partial t}$ (Radial Part)

$$\begin{aligned} \frac{r^2R'' + rR'}{R} + r^2\lambda^2 &= \nu^2 \\ r^2R'' + rR' + r^2\lambda^2 R &= \nu^2 R \\ r^2R'' + rR' + \left(\lambda^2 r^2 - \nu^2\right)R &= 0 \end{aligned}$$

and the last equation is called the **parametric Bessel equation**.