Separation of Variables – Eigenvalues of the Laplace Operator

Bernd Schröder

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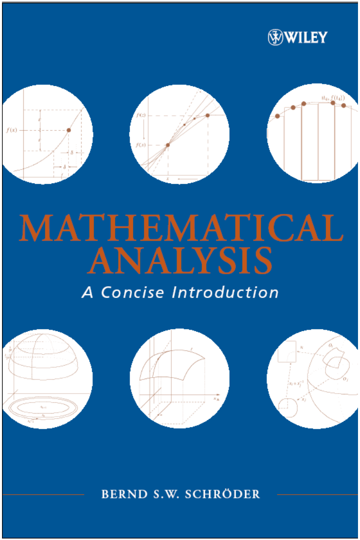
5. Solutions of the ordinary differential equations we obtain must typically be processed some more to give useful results for the partial differential equations.
6. Some very powerful and deep theorems can be used to formally justify the approach for many equations involving the Laplace operator.

How Deep?



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plus about 200 pages of really awesome functional analysis.

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