Separation of Variables – Legendre Equations

Bernd Schröder

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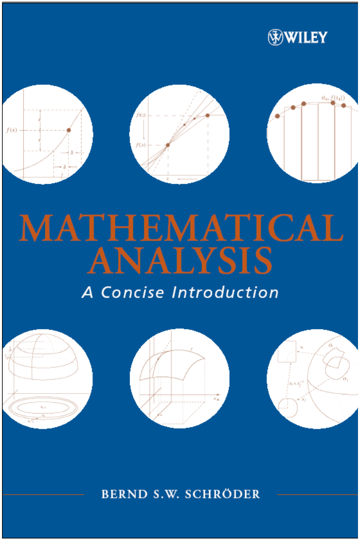
4. Key step: If $f(\rho) = g(\theta, \phi)$, then f and g must be constant.
5. Solutions of the ordinary differential equations we obtain must typically be processed some more to give useful results for the partial differential equations.
6. Some very powerful and deep theorems can be used to formally justify the approach for many equations involving the Laplace operator.

How Deep?



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plus about 200 pages of really awesome functional analysis.

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2. The time independent Schrödinger equation $-\frac{\hbar}{2m}\Delta\phi + V\phi = E\phi$ describes certain quantum mechanical systems, for example, the electron in a hydrogen atom. m is the mass of the electron, $\hbar = \frac{h}{2\pi}$, where h is Planck's constant, $V(\rho)$ is the electric potential and E is the energy eigenvalue.

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3. The equation $\Delta u = f(\rho)u$ had already been investigated in electrodynamics when its importance for the states of an electron in a hydrogen atom became clear.

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Both sides must be constant.

$\rho^2 f(\rho) - \rho^2\frac{R''}{R} - 2\rho\frac{R'}{R} = -\lambda$, or

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$\rho^2 R'' + 2\rho R' - \left(\lambda R + \rho^2 f(\rho)\right)R = 0.$

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$\rho^2 R'' + 2\rho R' - \left(\lambda R + \rho^2 f(\rho)\right) R = 0$. (QM: Laguerre polys.)

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$-\dfrac{T''}{T} = c$ leads to $T'' + cT = 0$.

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So the function T must be of the form $T(\theta) = c_1\cos(m\theta) + c_2\sin(m\theta)$.

Separating the Equation $\Delta u = f(\rho)u$ (Polar Part)

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This equation is complicated, because it involves trigonometric functions.

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It turns out that the substitution $z = \cos(\phi)$ will simplify the equation.

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&= \sin^2(\phi)\frac{d^2}{dz^2}P - \cos(\phi)\left(\frac{d}{dz}P\right)
\end{aligned}
$$

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\left(1-z^2\right)\frac{d^2P}{dz^2} - 2z\frac{dP}{dz} + \left(\lambda - \frac{m^2}{1-z^2}\right)P &
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Legendre Equations

Let λ be a real number and let m be a nonnegative integer. The differential equation

$$\left(1-x^2\right)y''-2xy'+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0$$

is called the **generalized Legendre equation**.

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m is a nonnegative integer, because this is required through the equation for $T(\theta)$ in the separation of variables.

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2. z approaching ± 1 corresponds to $\cos(\phi)$ approaching ± 1, which corresponds to ϕ approaching 0 and π.
3. So, physically this would mean that for $\lambda \neq l(l+1)$, the function u would be infinite on the z-axis, which is not sensible.