Separation of Variables – Oscillating Strings

Bernd Schröder

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6. For the equation in this presentation, Fourier series will allow us to get the actual solution of the problem.

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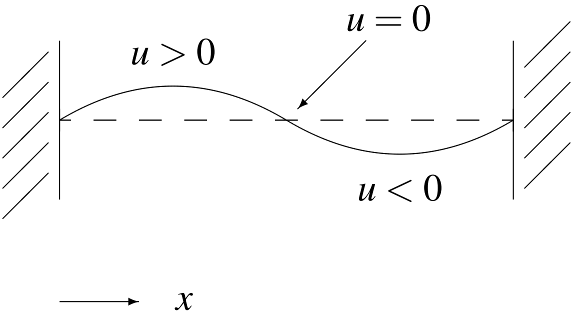

The Equation and the Initial and Boundary Conditions

1. $u(x,t)$ is the vertical displacement of the particle at position x on the string, at time t.
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3. Initial condition. For $0 \leq x \leq \pi$: $u(x,0) = f(x)$ (see next slide), $\frac{\partial}{\partial t}u(x,0) = 0$ (string is not moving as it is released).

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3. Initial condition. For $0 \leq x \leq \pi$: $u(x,0) = f(x)$ (see next slide), $\frac{\partial}{\partial t}u(x,0) = 0$ (string is not moving as it is released).
4. $u(0,t) = u(\pi,t) = 0$ for $t > 0$ (endpoints of the string are fixed).

The Initial Shape

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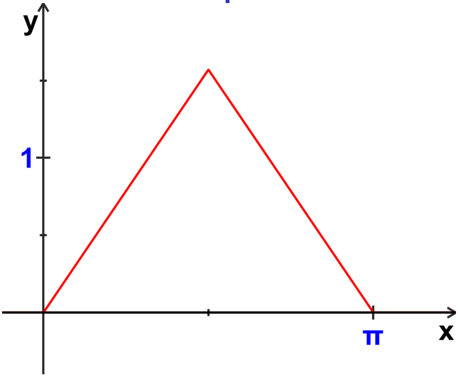

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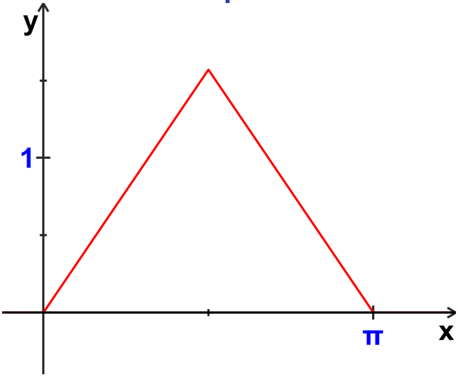

(We'll need to think about how to encode it.)

Separating the Equation

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$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t) \\ X(x)T''(t) &= X''(x)T(t)\end{aligned} \qquad u(x,t) := X(x)T(t)$$

Separating the Equation

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\frac{\partial^2}{\partial t^2}u(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t) \\
X(x)T''(t) &= X''(x)T(t) \\
\frac{T''(t)}{T(t)} &= \frac{X''(x)}{X(x)}
\end{aligned}$$

$$u(x,t) := X(x)T(t)$$

Separating the Equation

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$$\frac{T''(t)}{T(t)} = \lambda \quad \text{and} \quad \frac{X''(x)}{X(x)} = \lambda$$

$$T''(t) - \lambda T(t) = 0 \quad \text{and} \quad X''(x) - \lambda X(x) = 0$$

$$X''(x) - \lambda X(x) = 0, \; X(0) = X(\pi) = 0$$

$X''(x) - \lambda X(x) = 0,\ X(0) = X(\pi) = 0$

The solution is either $X(x) = k_1 e^{\mu x} + k_2 e^{-\mu x}$

$X''(x)-\lambda X(x)=0,\ X(0)=X(\pi)=0$

The solution is either $X(x)=k_1e^{\mu x}+k_2e^{-\mu x}$ or it is $X(x)=k_1\cos(\mu x)+k_2\sin(\mu x)$.

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$X(x) = k_1 e^{\mu x} + k_2 e^{-\mu x}$ is not possible because of the boundary conditions:

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0 = k_1\left(e^{\mu\pi}-e^{-\mu\pi}\right),
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which forces $k_1 = k_2 = 0$ and then $X(x) = 0$, which cannot be.
So $X(x) = k_1 \cos(\mu x) + k_2 \sin(\mu x)$ with $\mu = \sqrt{|\lambda|}$.

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which forces $k_1 = k_2 = 0$ and then $X(x) = 0$, which cannot be. So $X(x) = k_1 \cos(\mu x) + k_2 \sin(\mu x)$ with $\mu = \sqrt{|\lambda|}$. In some presentations, $-\lambda$ is used instead of λ, because the outcome of this computation is anticipated.

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$$0 \;=\; X(0) = k_1 \cdot 1 + k_2 \cdot 0$$

$X(x) = k_1 \cos(\mu x) + k_2 \sin(\mu x)$, $X(0) = X(\pi) = 0$

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\end{aligned}
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Requires μ to be a nonnegative integer n. (Animation.)

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So $X(x) = k_1 \sin(nx)$ and $\lambda = -\mu^2 = -n^2$.

Back to T

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$$\frac{T''}{T} = \lambda$$

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$$\begin{aligned} \frac{T''}{T} &= \lambda = -n^2 \\ T'' + n^2 T &= 0 \end{aligned}$$

Back to T

$$\begin{aligned}
\frac{T''}{T} &= \lambda = -n^2 \\
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T(t) &= c_1 \cos(nt) + c_2 \sin(nt)
\end{aligned}$$

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\frac{T''}{T} &= \lambda = -n^2 \\
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T'(t) &= -c_1 n \sin(nt) + c_2 n \cos(nt)
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0 &= -c_1 n \cdot 0 + c_2 n \cdot 1
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$$

So $T(t) = c_1 \cos(nt)$.

Back to u

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$$u(x,t) = X(x)T(t)$$

Back to u

$$
\begin{aligned}
u(x,t) &= X(x)T(t) \\
&= k_1 \sin(nx)
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Back to u

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= k_1 \sin(nx) c_1 \cos(nt) \\ u_n(x,t) &= b_n \sin(nx)\cos(nt) \end{aligned}$$

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None of these solutions fit our initial condition.

Back to u

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None of these solutions fit our initial condition. But because all u_n solve the equation, the boundary condition and the initial condition on the time derivative,

Back to u

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u(x,t) &= X(x)T(t) \\
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None of these solutions fit our initial condition. But because all u_n solve the equation, the boundary condition and the initial condition on the time derivative, so should

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) \cos(nt).$$

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$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos(nt).$$

(Significant theory required to assure the infinite summation does not destroy anything.)

A Trick to Allow the Use of Sine Series

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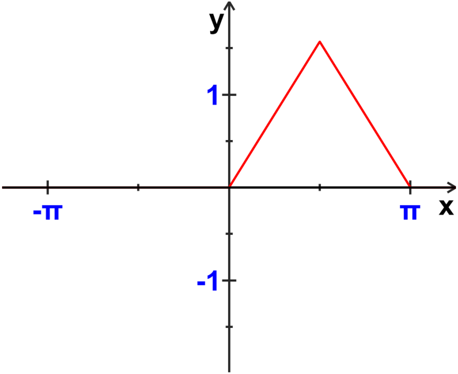
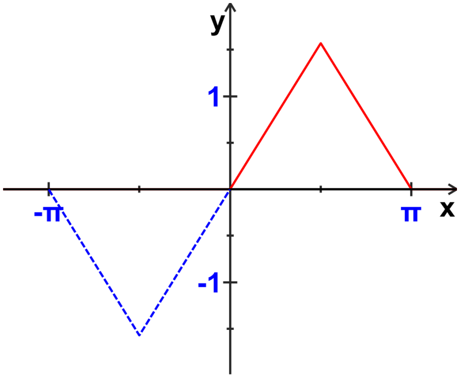
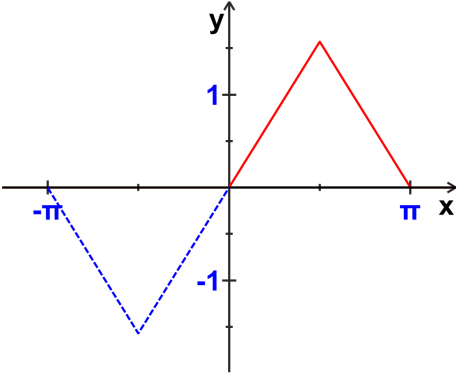


A Trick to Allow the Use of Sine Series



A Trick to Allow the Use of Sine Series



$$f(x) = \begin{cases} \frac{\pi}{2} - \left| x - \frac{\pi}{2} \right|; & \text{for } 0 \le x \le \pi, \\ \left| x + \frac{\pi}{2} \right| - \frac{\pi}{2}; & \text{for } -\pi \le x \le 0, \end{cases}$$

Fourier Coefficients

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

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$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}-\left|x-\frac{\pi}{2}\right|\right)\sin(nx)\,dx \end{aligned}$$

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because for even n, the integrand is odd with respect to the center $\frac{\pi}{2}$.

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because for even n, the integrand is odd with respect to the center $\frac{\pi}{2}$. For odd n, the integrand is even with respect to the center $\frac{\pi}{2}$ and we continue as follows.

Even/Odd With Respect to $\frac{\pi}{2}$

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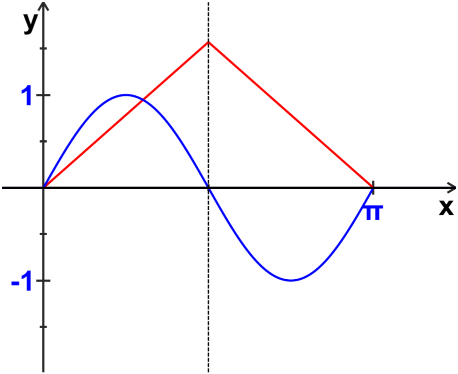
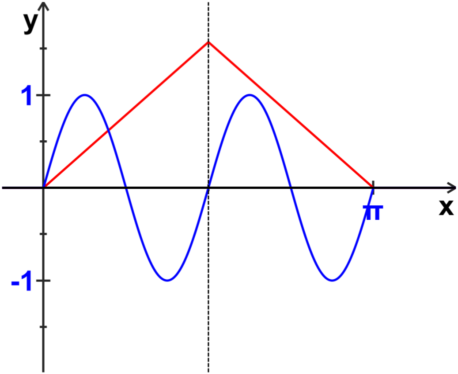


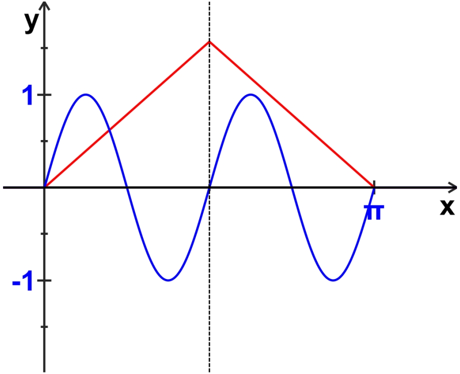
$f(x)\sin(2x)$

Even/Odd With Respect to $\frac{\pi}{2}$



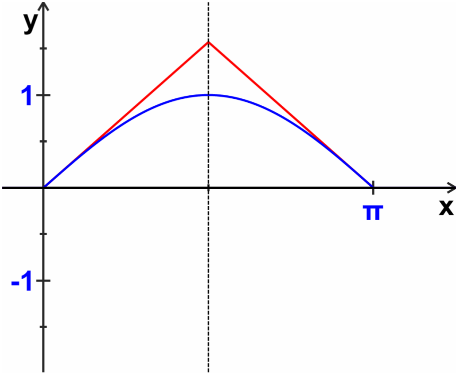
$f(x)\sin(4x)$

Even/Odd With Respect to $\frac{\pi}{2}$



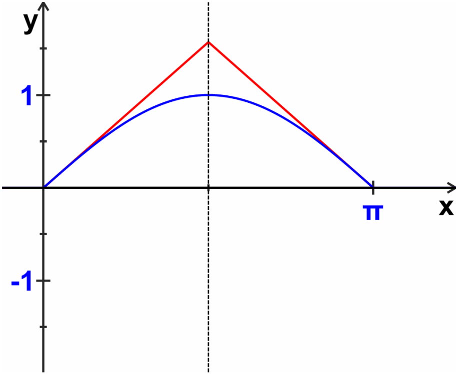
$f(x)\sin(4x)$
(and so on)

Even/Odd With Respect to $\frac{\pi}{2}$



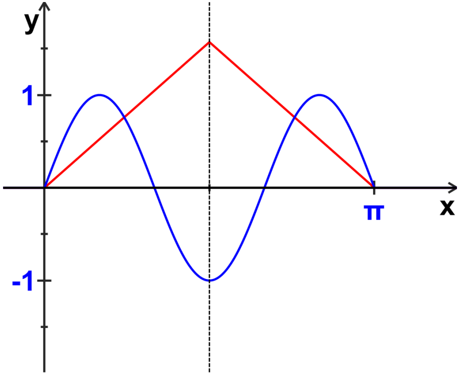
$f(x)\sin(x)$

Even/Odd With Respect to $\frac{\pi}{2}$



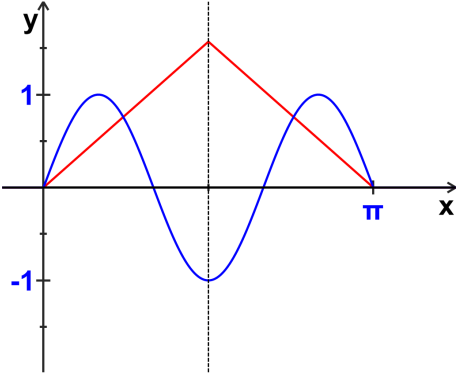
$f(x)\sin(x)$
(Doubles)

Even/Odd With Respect to $\frac{\pi}{2}$



$f(x)\sin(3x)$
(Doubles)

Even/Odd With Respect to $\frac{\pi}{2}$



$f(x)\sin(3x)$
(Doubles)
(and so on)

Fourier Coefficients

Fourier Coefficients

$$b_{2k+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|\right) \sin\left((2k+1)x\right)\, dx$$

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\begin{aligned}
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&= \frac{4}{\pi}\left[-x\frac{1}{2k+1}\cos\big((2k+1)x\big)\right|_0^{\frac{\pi}{2}}
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$$

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&= \frac{4}{\pi}\left[-x\frac{1}{2k+1}\cos\big((2k+1)x\big) + \frac{1}{(2k+1)^2}\sin\big((2k+1)x\big)\right]_0^{\frac{\pi}{2}} \\
&= -\frac{2}{2k+1}\cos\left((2k+1)\frac{\pi}{2}\right) + \frac{4}{\pi}\frac{1}{(2k+1)^2}\sin\left((2k+1)\frac{\pi}{2}\right)
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&= -\frac{2}{2k+1}\cos\left((2k+1)\frac{\pi}{2}\right) + \frac{4}{\pi}\frac{1}{(2k+1)^2}\sin\left((2k+1)\frac{\pi}{2}\right) \\
&= \frac{4}{\pi}\frac{(-1)^k}{(2k+1)^2}
\end{aligned}$$

The Solution

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$$u(x,t) := \sum_{k=0}^{\infty} \frac{4}{\pi} \frac{(-1)^k}{(2k+1)^2} \sin\big((2k+1)x\big) \cos\big((2k+1)t\big)$$

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But what does that look like?

The Analytical Solution (also animated)

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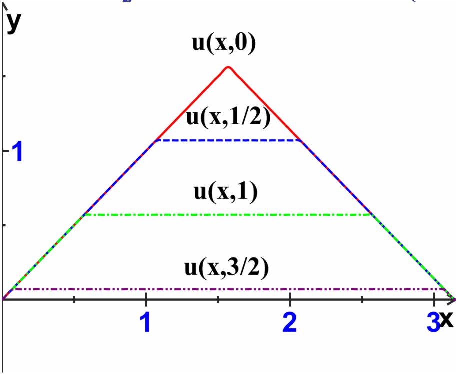
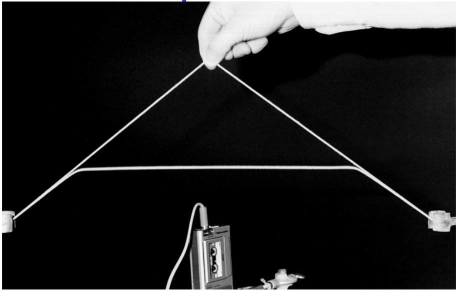


The Real Experiment

The Real Experiment



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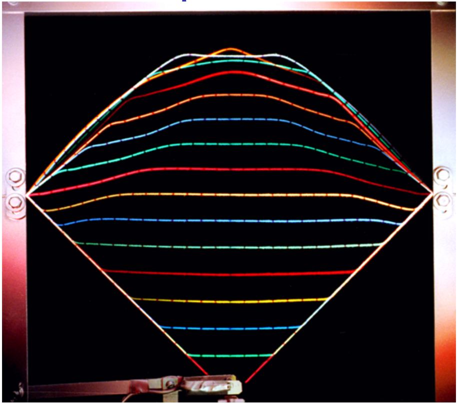
The Real Experiment



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