Series Solutions

Bernd Schröder

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That's it.

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That's it. (Except convergence analysis. Separate topic.)

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$$\begin{aligned} y'' - xy' + xy &= 0 \\ \left(\sum_{n=0}^{\infty} c_n x^n\right)'' - x\left(\sum_{n=0}^{\infty} c_n x^n\right)' + x\left(\sum_{n=0}^{\infty} c_n x^n\right) &= 0 \end{aligned}$$

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\begin{aligned}
y'' - xy' + xy &= 0 \\
\left(\sum_{n=0}^{\infty} c_n x^n \right)'' - x \sum_{n=1}^{\infty} c_n n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n &= 0
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Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$

$$\begin{aligned} y'' - xy' + xy &= 0 \\ \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n &= 0 \end{aligned}$$

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\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0
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$$k := n-2$$

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k := n-2 \qquad k := n \qquad k := n+1 &
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\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0
\end{aligned}
$$

$$k := n-2 \qquad k := n \qquad k := n+1$$

$$n = k+2 \qquad n = k$$

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\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
k := n-2 \qquad\quad k := n \qquad\quad k := n+1 & \\
n = k+2 \qquad\quad n = k \qquad\quad n = k-1 &
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\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0
\end{aligned}$$

$$k := n-2 \qquad k := n \qquad k := n+1$$

$$n = k+2 \qquad n = k \qquad n = k-1$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k$$

Solve the Initial Value Problem $y''-xy'+xy=0$, $y(0)=0$, $y'(0)=1$

$$
\begin{aligned}
y''-xy'+xy &= 0\\
\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n &= 0\\
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0\\
k:=n-2 \qquad\quad k:=n \qquad\quad k:=n+1 &\\
n=k+2 \qquad\quad n=k \qquad\quad n=k-1 &
\end{aligned}
$$

$$
\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k
$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$

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\begin{aligned}
y'' - xy' + xy &= 0 \\
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\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \\
k := n-2 \qquad k := n \qquad k := n+1 & \\
n = k+2 \qquad n = k \qquad n = k-1 & \\
\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k &= 0
\end{aligned}
$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k = 0$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k = 0$$

$$2c_2$$

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$$2c_2 + \sum_{k=1}^{\infty}x^k\Big[(k+2)(k+1)c_{k+2}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty}x^k\Big[(k+2)(k+1)c_{k+2} - kc_k$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty}x^k\Big[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\Big]$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$\begin{aligned}
\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k &= 0 \\
2c_2 + \sum_{k=1}^{\infty}x^k\left[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\right] &= 0
\end{aligned}$$

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\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k &= 0 \\
2c_2 + \sum_{k=1}^{\infty}x^k\left[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\right] &= 0
\end{aligned}$$

$$2c_2 = 0$$

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$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}kc_kx^k + \sum_{k=1}^{\infty}c_{k-1}x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty}x^k\left[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\right] = 0$$

$$2c_2 = 0, \qquad c_2 = 0$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$
\begin{aligned}
\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k &= 0 \\
2c_2 + \sum_{k=1}^{\infty} x^k \Big[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\Big] &= 0 \\
2c_2 = 0, \qquad c_2 = 0 & \\
(k+2)(k+1)c_{k+2} - kc_k + c_{k-1} &= 0
\end{aligned}
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2c_2 + \sum_{k=1}^{\infty}x^k\Big[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\Big] &= 0 \\
2c_2 = 0, \qquad c_2 = 0 & \\
(k+2)(k+1)c_{k+2} - kc_k + c_{k-1} &= 0 \\
(k+2)(k+1)c_{k+2} &= kc_k - c_{k-1}
\end{aligned}$$

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$$
\begin{aligned}
\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k &= 0 \\
2c_2 + \sum_{k=1}^{\infty} x^k \left[(k+2)(k+1)c_{k+2} - kc_k + c_{k-1}\right] &= 0 \\
2c_2 = 0, \qquad c_2 = 0 & \\
(k+2)(k+1)c_{k+2} - kc_k + c_{k-1} &= 0 \\
(k+2)(k+1)c_{k+2} &= kc_k - c_{k-1} \\
c_{k+2} &= \frac{kc_k - c_{k-1}}{(k+2)(k+1)}
\end{aligned}
$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$c_0 = y(0) = 0,$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1,$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0,$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

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$$(k=1) \quad c_3 = c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)}$$

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$$(k=1) \quad c_3 \quad = \quad c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

$$\begin{aligned}
(k=1) \quad c_3 &= c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6} \\
(k=2) \quad c_4 &= c_{2+2} = \frac{2 \cdot c_2 - c_1}{(2+2)(2+1)}
\end{aligned}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

$$(k=1) \quad c_3 = c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6}$$

$$(k=2) \quad c_4 = c_{2+2} = \frac{2 \cdot c_2 - c_1}{(2+2)(2+1)} = -\frac{1}{12}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

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$$(k=2) \quad c_4 = c_{2+2} = \frac{2 \cdot c_2 - c_1}{(2+2)(2+1)} = -\frac{1}{12}$$

$$(k=3) \quad c_5 = c_{2+3} = \frac{3 \cdot c_3 - c_2}{(3+2)(3+1)}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

$$(k=1) \quad c_3 = c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6}$$

$$(k=2) \quad c_4 = c_{2+2} = \frac{2 \cdot c_2 - c_1}{(2+2)(2+1)} = -\frac{1}{12}$$

$$(k=3) \quad c_5 = c_{2+3} = \frac{3 \cdot c_3 - c_2}{(3+2)(3+1)} = \frac{3 \cdot \frac{1}{6} - 0}{20}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

$$(k=1) \quad c_3 = c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6}$$

$$(k=2) \quad c_4 = c_{2+2} = \frac{2 \cdot c_2 - c_1}{(2+2)(2+1)} = -\frac{1}{12}$$

$$(k=3) \quad c_5 = c_{2+3} = \frac{3 \cdot c_3 - c_2}{(3+2)(3+1)} = \frac{3 \cdot \frac{1}{6} - 0}{20} = \frac{1}{40}$$

Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$ (continued)

$$c_0 = y(0) = 0, \quad c_1 = y'(0) = 1, \quad c_2 = 0, \quad c_{k+2} = \frac{kc_k - c_{k-1}}{(k+2)(k+1)}$$

$$(k=1) \quad c_3 = c_{1+2} = \frac{1 \cdot c_1 - c_0}{(1+2)(1+1)} = \frac{1}{6}$$

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Solve the Initial Value Problem $y'' - xy' + xy = 0$, $y(0) = 0$, $y'(0) = 1$

1. The series expansion of the solution starts with the polynomial $y(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{1}{60}x^6$.

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2. It should be noted that the above polynomial is *<u>not</u>* the solution itself.
3. To emphasize this fact, the solution can be stated as $y(x) = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{1}{60}x^6 + \cdots$.

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4. Because the computations are quite tedious, use a computer algebra system.

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(Remember that we went up to $k = 4$, so terms up to order 4 *must* cancel.)