The Radius of Convergence of a Series Solution

Bernd Schröder

Direct Computation of the Radius of Convergence May Not be Possible, But ...

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1. A function f is called **analytic at** x_0 if and only if f equals its Taylor series expansion in some open interval about x_0.

   That is, there is an $\varepsilon > 0$ such that $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$

   for all x with $|x - x_0| < \varepsilon$.

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2. The same definition works for a function of a complex variable, and we will need to mind complex numbers throughout.
3. For the differential equation $y'' + P(x)y' + Q(x)y = 0$ the point x_0 is called an **ordinary point** if and only if both P and Q are analytic at x_0. A point that is not ordinary will be called a **singular point**.

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4. If x_0 is an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

then there exist two linearly independent solutions of the equation that are power series about x_0. That is, there are two linearly independent solutions of the form $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$. Moreover, the radius of convergence of the power series is at least the distance from x_0 to the closest singular point in the complex plane.

Lower Bound for the Radius of Convergence of Solutions of $\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$ about $x_0 = 0$ and $x_0 = 2$

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$$\begin{aligned} \left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y &= 0 \\ y'' + \frac{3x}{1+x^2} y' + \frac{2}{\left(1+x^2\right)^2} y &= 0 \end{aligned}$$

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Singular points at $\pm i$.

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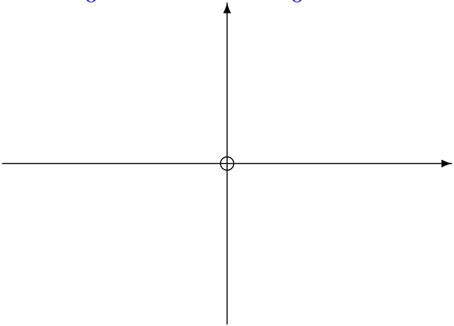

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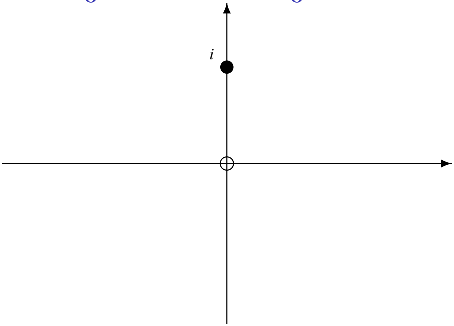

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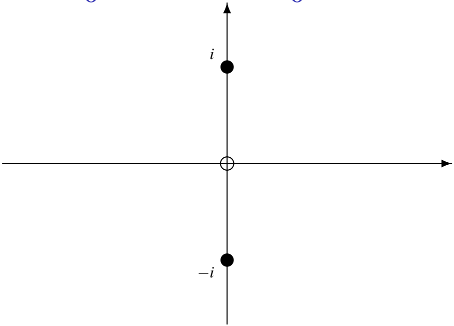

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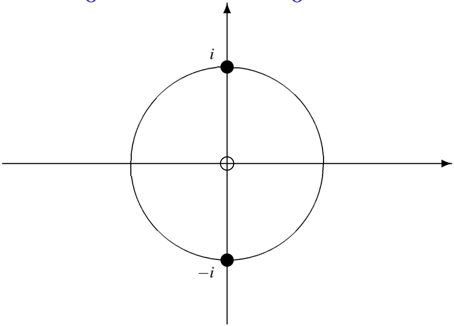

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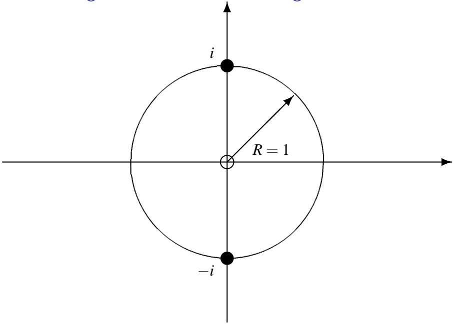

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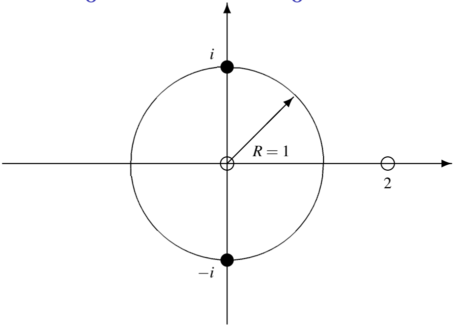

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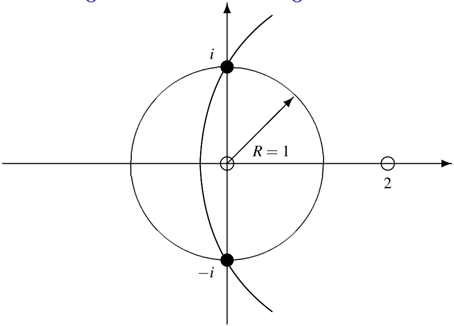

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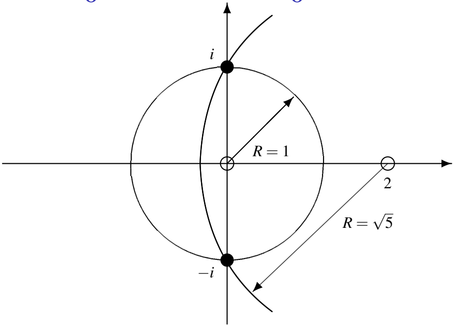

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$

$f(x) = \frac{1}{1+x^2}$ Solves

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$$y(x) := \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right)y' + 2y = 0$$

$$y(x) := \frac{1}{1+x^2}$$

$$\left(1+x^2\right)^2 \cdot \frac{d^2}{dx^2}y(x) + 3x \cdot \left(1+x^2\right) \cdot \frac{d}{dx}y(x) + 2\,y(x) \text{ simplify } \rightarrow 0$$

$f(x) = \frac{1}{1+x^2}$ Solves

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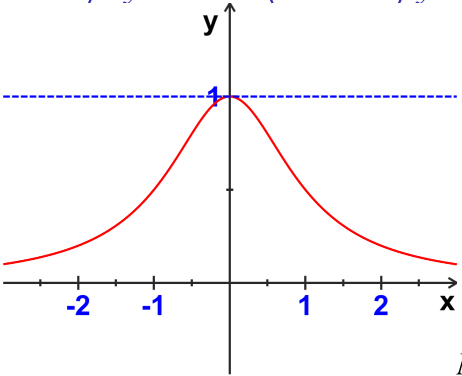
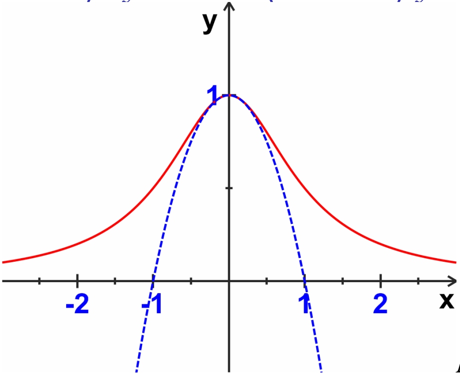


$N = 0$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

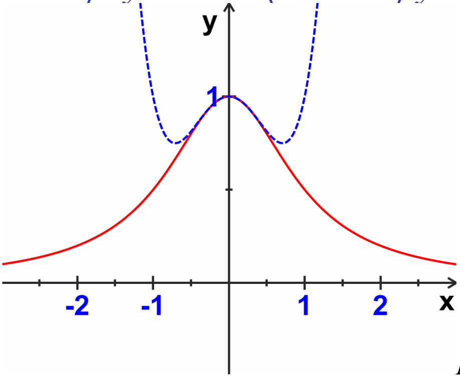
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 2$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

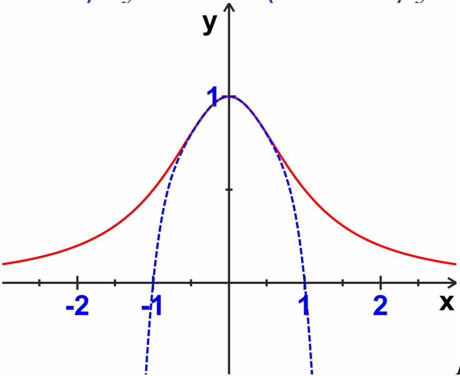
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 4$

$f(x) = \frac{1}{1+x^2}$ Solves

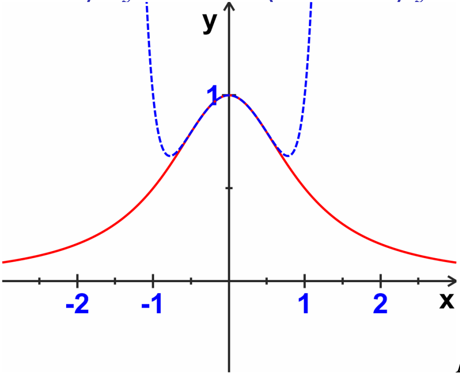
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 6$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

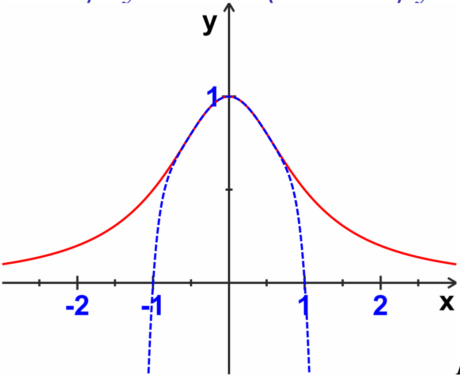
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 8$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

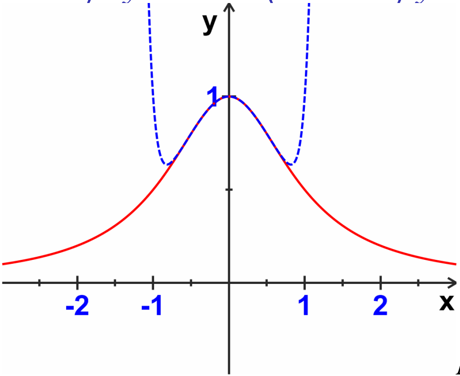
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 10$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

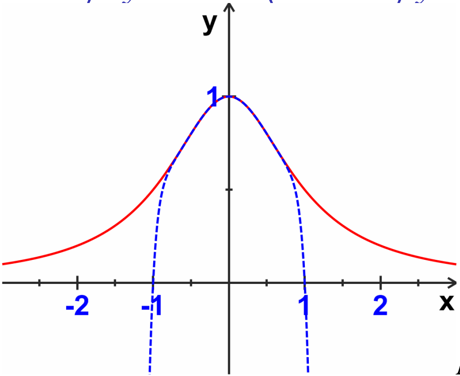
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 12$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

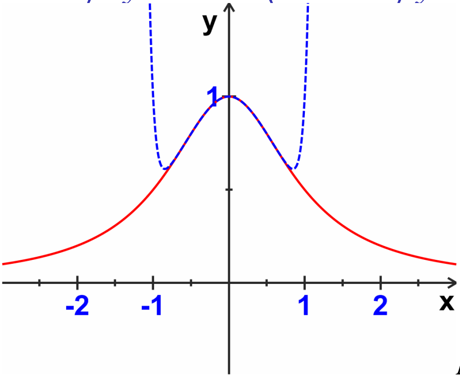
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 14$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

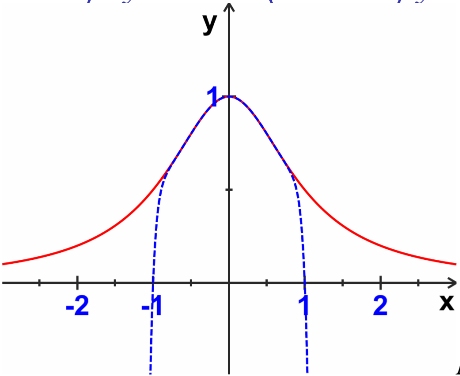
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 16$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

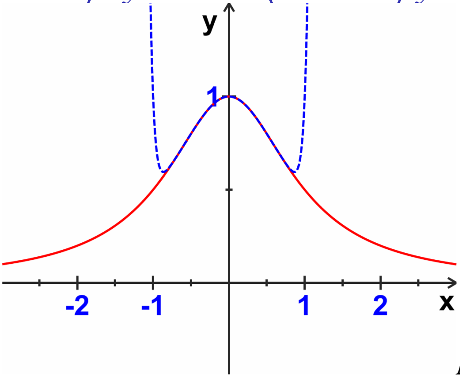
$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 18$

$$f(x) = \frac{1}{1+x^2} \text{ Solves}$$

$$\left(1+x^2\right)^2 y'' + 3x\left(1+x^2\right) y' + 2y = 0$$



$N = 20$