Theory of Linear Ordinary Differential Equations

Bernd Schröder

Definition

A **linear n-th order differential equation** is of the form

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x),$$

with a_n not being the constant function 0.

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Note that the coefficients are functions. The results in this presentation apply to constant coefficient equations as well as Cauchy-Euler equations or the equations that are being solved with series solutions.

Existence and Uniqueness

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Every initial value problem of the form

$$\begin{aligned} a_n(x)y^{(n)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x) &= g(x), \\ y(x_0) &= y_0, \\ y'(x_0) &= y_1, \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1}, \end{aligned}$$

where a_n is not the constant function 0 and all $a_i(x)$ and $g(x)$ have continuous first derivatives, has a unique solution.

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where a_n is not the constant function 0 and all $a_i(x)$ and $g(x)$ have continuous first derivatives, has a unique solution.

So, in some ways, the solutions look like n-dimensional space.

We are interested in using this analogy.

Proof of the Existence and Uniqueness Theorem

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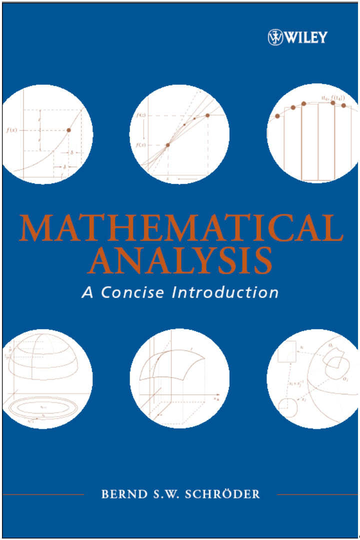

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Furthering the Analogy Between Vectors and Solutions

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Superposition Principle. Let y_1 and y_2 be solutions of the homogeneous linear differential equation

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=0,$$

and let c_1 and c_2 be real numbers. Then $y(x):=c_1y_1(x)+c_2y_2(x)$ is a solution, too.

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So solutions of homogeneous equations have the same algebraic properties as vectors.

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$$c_1\Big(\qquad a_n(x)y_1^{(n)}(x) \quad +\cdots \quad + \qquad a_0(x)y_1(x) \quad \Big) = \quad c_1\cdot 0$$

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Handling Inhomogeneous Equations

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For the linear inhomogeneous differential equation

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let $y_h(x)$ denote the general solution of the corresponding homogeneous equation. Moreover let $y_p(x)$ be one particular solution of the inhomogeneous equation. Then the general solution of the inhomogeneous equation is

$$y(x)=y_p(x)+y_h(x).$$

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So the theory of inhomogeneous equations is pretty much reduced to that of homogeneous equations.

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\begin{array}{rcccccc}
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\hline
\end{array}
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$$\begin{array}{rcl} a_n(x)y_p^{(n)}(x) + \cdots + a_0(x)y_p(x) &=& g(x) \\ +\big(a_n(x)y_h^{(n)}(x) + \cdots + a_0(x)y_h(x) &=& 0 \quad \big) \\ \hline a_n(x)(y_p+y_h)^{(n)}(x) + \cdots + a_0(x)(y_p+y_h)(x) &=& g(x) \end{array}$$

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Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors. Then any sum

$$\sum_{i=1}^{n} c_i \vec{v}_i = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$$

with the c_i being real numbers is called a **linear combination** of the vectors.

Linear Independence for Vectors

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A set of n vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called **linearly dependent** if and only if there are numbers $c_1, \ldots, c_n$, which are not all zero, such that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$, where $\vec{0}$ denotes the **null vector**, for which all components are zero.

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If no such numbers exist, the set of vectors is called **linearly independent**. That is, a set of n vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is called **linearly independent** if and only if the *only* numbers $c_1, \cdots, c_n$, for which $\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$ are $c_1 = c_2 = \cdots = c_n = 0$.

Determine if the vectors $(1,1,3)$, $(2,4,2)$ and $(3,-1,4)$ are linearly independent.

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$$c_1\begin{pmatrix}1\\1\\3\end{pmatrix}+c_2\begin{pmatrix}2\\4\\2\end{pmatrix}+c_3\begin{pmatrix}3\\-1\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

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$0 = c_3 = c_2 = c_1$, and the vectors are linearly independent.

Determine if $\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$ are linearly independent.

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$$2c_1 \quad - \quad 1c_2 \quad + \quad 3c_3 \quad = \quad 0$$

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$$\begin{aligned} 2c_1 - 1c_2 + 3c_3 &= 0 \\ -2c_1 + 2c_2 - 2c_3 &= 0 \end{aligned}$$

Determine if $\begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$ are linearly independent.

$$\begin{array}{rcrcrcl} 2c_1 & - & 1c_2 & + & 3c_3 & = & 0 \\ -2c_1 & + & 2c_2 & - & 2c_3 & = & 0 \\ -4c_1 & + & 3c_2 & - & 5c_3 & = & 0 \end{array}$$

$$
\begin{array}{rcrcrcl}
2c_1 & - & c_2 & + & 3c_3 & = & 0 \\
-2c_1 & + & 2c_2 & - & 2c_3 & = & 0 \\
-4c_1 & + & 3c_2 & - & 5c_3 & = & 0
\end{array}
$$

$$\begin{array}{rcrcrcr} 2c_1 & - & c_2 & + & 3c_3 & = & 0 \\ -2c_1 & + & 2c_2 & - & 2c_3 & = & 0 \\ -4c_1 & + & 3c_2 & - & 5c_3 & = & 0 \end{array}$$

$$2c_1 - c_2 + 3c_3 = 0$$

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$c_2 = -c_3$

$$\begin{array}{rcrcrcl} 2c_1 & - & c_2 & + & 3c_3 & = & 0 \\ -2c_1 & + & 2c_2 & - & 2c_3 & = & 0 \\ -4c_1 & + & 3c_2 & - & 5c_3 & = & 0 \end{array}$$

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$c_2 = -c_3$, $c_1 = \dfrac{c_2 - 3c_3}{2}$

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choose $c_3 = 1$

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Why use Matrices?

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$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 4 \end{pmatrix}$$

Matrices

Matrices

Let m and n be positive integers. An $m \times n$-**matrix** is a rectangular array of mn numbers a_{ij}, commonly indexed and written as follows.

$$A = (a_{i,j})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3(n-1)} & a_{3n} \\ \vdots & & & & \vdots \\ a_{(m-1)1} & a_{(m-1)2} & \cdots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{pmatrix}$$

The index i is called the **row index** and the index j is called the **column index**.

Determinants

Determinants

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix. Then we define the **determinant** of A to be

$$\det(A) := \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| := a_{11}a_{22} - a_{12}a_{21}.$$

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Let $A = (a_{ij})_{i,j=1,\ldots,n}$ be a square matrix and let A_{ij} be the matrix obtained by erasing the i^{th} row and the j^{th} column. Then the **determinant** of A is defined recursively by

$$\det(A) := |A| := \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(A_{ij}\right) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det\left(A_{ij}\right),$$

where the i in the first sum is an arbitrary row and the j in the second sum is an arbitrary column.

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1. The determinant gives the n-dimensional volume of the parallelepiped spanned by the column vectors.
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3. Computation of characteristic polynomials.

Determine if $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ are linearly independent.

Determine if $\begin{pmatrix}1\\1\\3\end{pmatrix}$, $\begin{pmatrix}2\\4\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\\4\end{pmatrix}$ are linearly independent.

$$\det\begin{pmatrix}1 & 2 & 3\\ 1 & 4 & -1\\ 3 & 2 & 4\end{pmatrix}$$

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$$= 1\cdot 18$$

Determine if $\begin{pmatrix}1\\1\\3\end{pmatrix}$, $\begin{pmatrix}2\\4\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\\4\end{pmatrix}$ are linearly independent.

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$$= 1\cdot\det\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix} - 1\cdot\det\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix} + 3\cdot\det\begin{pmatrix}2 & 3\\ 4 & -1\end{pmatrix}$$
$$= 1\cdot 18 - 1\cdot 2$$

Determine if $\begin{pmatrix}1\\1\\3\end{pmatrix}$, $\begin{pmatrix}2\\4\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\\4\end{pmatrix}$ are linearly independent.

$$\det\begin{pmatrix}1 & 2 & 3\\ 1 & 4 & -1\\ 3 & 2 & 4\end{pmatrix}$$

$$= 1\cdot\det\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix} - 1\cdot\det\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix} + 3\cdot\det\begin{pmatrix}2 & 3\\ 4 & -1\end{pmatrix}$$
$$= 1\cdot 18 - 1\cdot 2 + 3\cdot(-14)$$

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$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 4 \end{pmatrix}$$

$$\begin{aligned} &= 1 \cdot \det \begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \\ &= 1 \cdot 18 - 1 \cdot 2 + 3 \cdot (-14) \\ &= -26 \end{aligned}$$

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$$\begin{aligned}
&= 1\cdot\det\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix} - 1\cdot\det\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix} + 3\cdot\det\begin{pmatrix}2 & 3\\ 4 & -1\end{pmatrix}\\
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The vectors are linearly independent.

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$$\det \begin{pmatrix} 2 & -1 & 3 \\ -2 & 2 & -2 \\ -4 & 3 & -5 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 2 & -2 \\ 3 & -5 \end{pmatrix}$$

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&\quad +(-4)\cdot\det\begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \\
&= 2\cdot(-4)-(-2)(-4)+(-4)(-4) \\
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\end{aligned}$$

The vectors are linearly dependent.

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Otherwise we would not be able to recognize that a family like $y_{c_1,c_2}(x) = c_1 \sin^2(x) + c_2\big(1-\cos(2x)\big)$ is *not* the general solution of $\sin(x)y'' - \cos(x)y' + 2\sin(x)y = 0$.

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(The family has only one constant, because $2\sin^2(x) = 1-\cos(2x)$.)

Let $f_1, f_2, \ldots, f_n$ be functions. Then any sum

$$\sum_{i=1}^{n} c_i f_i = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

with the c_i being real numbers is called a **linear combination** of the functions.

Linear Independence for Functions

Linear Independence for Functions

A set of n functions $\{f_1,\ldots,f_n\}$ is called **linearly dependent** if and only if there are numbers $c_1,\ldots,c_n$, which are not all zero, such that $c_1f_1+\cdots+c_nf_n=0$. That is, $c_1,\ldots,c_n$ must be such that for *all* x in the domain of $f_1,\ldots,f_n$ we have $c_1f_1(x)+\cdots+c_nf_n(x)=0$.

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If no such numbers exist, then the set of functions is called **linearly independent**. That is, a set of n functions $\{f_1, \ldots, f_n\}$ is called **linearly independent** if and only if the only numbers $c_1, \ldots, c_n$, for which $\sum_{i=1}^{n} c_i f_i = 0$ are $c_1 = c_2 = \cdots = c_n = 0$.

The Wronskian

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Let $f_1, \cdots, f_n$ be $(n-1)$ times differentiable functions. If the **Wronskian**

$$W(f_1, \cdots, f_n)(x) := \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

is not equal to zero for some value of x, then $\{f_1, \cdots, f_n\}$ is a linearly independent set of functions.

Determine if t, e^t and te^t are linearly independent.

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$$\det\begin{pmatrix} t & e^t & te^t \\ 1 & e^t & te^t+e^t \\ 0 & e^t & te^t+2e^t \end{pmatrix}$$

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$$+0\cdot\det\begin{pmatrix} e^t & te^t \\ e^t & te^t+e^t \end{pmatrix}$$

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$$= t\left(te^{2t}+2e^{2t}-te^{2t}-e^{2t}\right)$$

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\end{aligned}$$

The functions are linearly independent.

Defining the General Solution

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The **general solution** of a differential equation is a family of functions so that for every initial value problem for the differential equation there is a unique choice of the coefficients that gives the solution of the initial value problem. A **particular solution** of a differential equation is *one* specific solution.

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In the theory, we typically work with initial value problems, because even this definition is a bit messy.

Solution Theorem for Linear Homogeneous Differential Equations

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The general solution of a linear homogeneous differential equation

$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=0,$

is of the form

$$y(x)=c_1y_1(x)+\cdots+c_ny_n(x),$$

where $\{y_1,\cdots,y_n\}$ is a linearly independent set of particular solutions of the linear homogeneous differential equation.